Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 4

1. Let V be a real vector space of dimension n and $\omega \in \Lambda^2(V^*)$ a 2-form, i.e. an alternating bilinear form $V \times V \to \mathbb{R}$. The matrix of ω in a basis $e_1, ..., e_n$ of V is $\omega_{ij} = \omega(e_i, e_j)$.

a) Let V be 2-dimensional and ω not identical to 0. Show that there exists a basis e_1, e_2 of V such that the matrix of ω has the form

$$\omega_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

b) Let V have arbitrary dimension n. Show that there exists a basis $e_1, ..., e_n$ of V and a number k with $2k \leq n$ such that

$$\omega = \sum_{i=1}^{k} \alpha_{2i-1} \wedge \alpha_{2i},$$

where $\alpha_1, ..., \alpha_n$ denote the dual basis of 1-forms. (For the proof, you can consider the subspace $W' = \{v \in V \mid \omega(v, w) = 0 \quad \forall w \in W\}$ for a subspace W of V.)

2. Let V be an n-dimensional vector space. A 2-form $\omega \in \Lambda^2(V^*)$ is called *decomposable* if there exist 1-forms α, β such that $\omega = \alpha \wedge \beta$. Using Ex.1, show that ω is decomposable if and only if $\omega \wedge \omega = 0$. If the dimension $n \geq 4$, find a 2-form which is not decomposable.

3. Let M be smooth manifold and α a differential form. Show that

$$i_{[X,Y]}\alpha = L_X i_Y \alpha - i_Y L_X \alpha,$$

for all vector fields X, Y.

4. Let M be a smooth manifold and $\omega \in \Omega^2(M)$ a 2-form. Show that for vector fields X, Y, Z

$$\begin{aligned} d\omega(X,Y,Z) &= L_X\omega(Y,Z) + L_Y\omega(Z,X) + L_Z\omega(X,Y) \\ &\quad -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y). \end{aligned}$$

Please hand in your solutions on Thursday, the 20th of November, in the lecture.