Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 3

1. a) Let (U, ϕ) be a chart of an n-dimensional differentiable manifold M. The coordinates of a point $p \in U$ are $(x_1(p), ..., x_n(p)) = \phi(p) \in \mathbb{R}^n$. Let $e_1, ..., e_n$ denote the *coordinate vector fields* $e_i = \partial/\partial x_i$, defined by their actions on smooth functions on U. Show that $e_1, ..., e_n$ form for each $p \in U$ a basis of the tangent space T_pM and that $[e_i, e_j] = 0$ for all i, j.

b) Let M be a differentiable manifold and $f : M \to \mathbb{R}$ a smooth function. Show that

$$[X, fY] = (L_X f)Y + f[X, Y],$$

for all vector fields X, Y on M. Calculate as an example the commutator of the vector fields $V = x^2ye_1 + e_2$ and $W = xe_1 - y^2e_2$ on $\mathbb{R}^2$ with coordinates (x,y) and $e_1 = \partial/\partial x, e_2 = \partial/\partial y$.

2. Let M be a differentiable manifold and X, Y complete vector fields. Denote the flows by Φ_t^X and Φ_t^Y, respectively. Show that the commutator vanishes if and only if the flows commute, i.e.

$$[X, Y] \equiv 0 \Longleftrightarrow \Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X \quad \forall t, s \in \mathbb{R}.$$

(To prove implication '$\Rightarrow$' you can consider for fixed s and p the curves

$$\alpha(t) = (\Phi_t^X \circ \Phi_s^Y)(p) \quad \text{and} \quad \beta(t) = (\Phi_s^Y \circ \Phi_t^X)(p)$$

through the point $\Phi_s^Y(p)$.)

3. Let M be an n-dimensional differentiable manifold. Suppose there are vector fields $e_1, ..., e_n$ defined in the neighbourhood of a point $p \in M$ which are everywhere linearly independent and such that all commutators $[e_i, e_j]$ vanish identically. Show that there is a chart (U, ϕ) around p such that $e_1, ..., e_n$ restricted to U are the associated coordinate vector fields.

(please turn)

4. Recall the definition of projective spaces from the last problem sheet. Show that $\mathbb{C}P^1$ is diffeomorphic to S^2. (Similarly, one can show that $\mathbb{R}P^1$ is diffeomorphic to S^1. The higher projective spaces are not diffeomorphic to spheres.)

Remark: The canonical projection $\pi : S^{2n+1} \to \mathbb{C}P^n$, in particular the case $n = 1$ where $\pi : S^3 \to S^2$, is called the *Hopf fibration.*

Please hand in your solutions on Thursday, the 13th of November, in the lecture.