Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 9

1. Let M, N be differentiable manifolds. Recall that a smooth map $\phi : M \to N$ is a *covering* if ϕ is surjective and for every $p \in N$ there exists a neighbourhood V such that $\phi^{-1}(V) = \dot{\bigcup}_{i\in I} U_i$ and $\phi : U_i \to V$ is a diffeomorphism. Prove that if M is compact, N connected and ϕ a local diffeomorphism, then ϕ is a covering.

2. Let (M, g) be an n-dimensional Riemannian manifold, f a smooth function and $p \in M$. Consider an orthonormal basis $\{e_i\}$ in T_pM and geodesics γ_i with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = e_i$. Prove the following formula:

$$(\Delta f)(p) = \sum_{i=1}^{n} \frac{d^2}{dt^2}|_{t=0} f(\gamma_i(t)).$$

3. Let (M, g) be a closed oriented Riemannian manifold and $\Delta_H = d^*d$ the Hodge-Laplace operator on functions (note that $\Delta_H f = -\Delta f = -div\,\nabla f$). A smooth function f is called *eigenfunction* of Δ_H with eigenvalue $\lambda \in \mathbb{R}$ if

$$\Delta_H f = \lambda f, \quad f \not\equiv 0.$$

a) Show that the eigenvalues of Δ_H are non-negative.

b) Let $M = S^n$ with the standard metric. Fix a vector $v \in \mathbb{R}^{n+1} \setminus 0$ and define the smooth function $f_v : S^n \to \mathbb{R}, p \mapsto \langle p, v\rangle$, where $\langle , \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^{n+1}$. Prove that f_v is an eigenfunction of Δ_H on S^n with eigenvalue n.

4. Let (M, g) be a closed oriented n-dimensional Riemannian manifold. Prove the following theorem of Lichnerowicz: if the Ricci curvature of M is greater or equal to $k > 0$ (i.e. $Ric(X) \geq k|X|^2$ for all tangent vectors X) then the non-zero eigenvalues of Δ_H are bounded from below by nk. Show that this bound is optimal for S^n with the standard metric.

Hint: Consider the function $h = \frac{1}{2}|\nabla f|^2$ and use a Bochner formula for Δh from the lecture. Show that $|\nabla(\nabla f)|^2 \geq (\Delta f)^2/n$.

Please hand in your solutions on Monday, the 5th of July, in the lecture.