Mathematisches Institut
Universität München

Mark Hamilton
Office hour: Mon 1-2 pm, Room 308 SS 04

## Geometry of Manifolds II (Prof. Kotschick)
## Problem sheet 7

1. a) Let $N$ be a closed manifold. Show that $N \times S^2$ admits a metric of positive scalar curvature.

b) Find a closed 4-manifold $X$ which admits a metric with $scal > 0$ but which does not admit a metric with $Ric \geq 0$.

2. Let $(M, g)$ be a Riemannian manifold and $\omega \in \Omega^*(M)$ a differential form. Fix a point $p \in M$, an orthonormal basis $\{X_i\}$ of $T_pM$ with dual basis $\{\theta_i\}$ and let $\lambda_\alpha$ be the eigenvalues of the curvature operator $\mathfrak{R}_p : \Lambda^2 T_pM \to \Lambda^2 T_pM$ with $\Theta_\alpha \in \Lambda^2 T_p^*M$ the duals of the corresponding eigenvectors. Show that

$$\sum_{i,j=1}^{n} g([\theta_i \cdot \theta_j, R(X_i, X_j)\omega], \omega) = -\sum_\alpha \lambda_\alpha |[\Theta_\alpha, \omega]|^2.$$

Hint: Prove first that the left hand side equals

$$-\sum_{i<j,k<l}^{n} g(\mathfrak{R}(X_i \wedge X_j), X_k \wedge X_l) g([\theta_k \cdot \theta_l, \omega], [\theta_i \cdot \theta_j, \omega])$$

and then show that this expression is independent of the orthonormal basis of $\Lambda^2 T_pM$.

3. Let $(M^n, g)$ be a Riemannian manifold and $\omega$ a differential form of degree $k$ where $1 \leq k \leq n-1$. With the notation as in Ex. 2 prove that each of the following two conditions implies that $\omega$ vanishes identically:

a) $[\theta_i \cdot \theta_j, \omega] = 0$ for all $i, j$.

b) $[\Theta_\alpha, \omega] = 0$ for all $\alpha$.

4. Let $(X, g)$ be a closed oriented Riemannian 4-manifold. Prove the following Weitzenböck formula for 2-forms $\omega \in \Omega^2(X)$:

$$\Delta_H \omega = \nabla^* \nabla \omega - 2\, W(\omega) + \frac{1}{3} scal\, \omega.$$

Here $W$ denotes the *Weyl tensor* which acts on 2-forms via

$$W(\omega) = W(\sum_{a,b=1}^{4} \omega_{ab}\theta_a \wedge \theta_b) = \frac{1}{2} \sum_{a,b,i,k=1}^{4} W_{abik}\omega_{ab}\theta_i \wedge \theta_k$$

and has components

$$W_{ijkl} = R_{ijkl} + \frac{1}{2}(\delta_{li}R_{kj} - \delta_{lj}R_{ki} - \delta_{ki}R_{lj} + \delta_{kj}R_{li}) - \frac{1}{6}(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki})scal,$$

where

$$\begin{aligned} R_{ijkl} &= g(R(E_i, E_j)E_k, E_l) \\ R_{ij} &= \sum_k R_{ikjk} \\ scal &= \sum_k R_{kk}, \end{aligned}$$

in a local orthonormal frame $\{E_i\}$ with dual frame $\{\theta_i\}$.

Hint: Prove first the identity

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1}^{4} \theta_i \cdot \theta_j \cdot R(E_i, E_j)(\theta_a \cdot \theta_b) &= \sum_{i=1}^{4}(-R_{ia}\theta_i \cdot \theta_b + R_{ib}\theta_i \cdot \theta_a) \\ &\quad + \sum_{i,k=1}^{4} (R_{ibak} - R_{iabk})\theta_i \cdot \theta_k. \end{aligned}$$

Please hand in your solutions on Monday, the 21st of June, in the lecture.