Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308
SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 6

1. Let (M, g) be an oriented Riemannian manifold and X a vector field. Use a local orthonormal frame which is parallel in one point to show that

$$L_X dvol_g = div(X) dvol_g.$$

2. Let (M, g) be a closed oriented Riemannian manifold and X a Killing field. Show that:
   a) The flow of X operates through isometries.
   b) The adjoint L_X^* of the map $L_X : \Omega^k(M) \to \Omega^k(M)$ with respect to the L^2-norm is $-L_X$.
   c) If $\omega \in \Omega^k(M)$ is harmonic then $L_X \omega = 0$.

3. Let (M, g) be a closed connected oriented Riemannian manifold.
   a) Show that for smooth functions f and h

$$\Delta(fh) = (\Delta f)h + f(\Delta h) + 2g(grad\, f, grad\, h).$$

Suppose that f is a smooth function with $\Delta f \geq 0$. Use Stokes' theorem to prove (without using Hodge theory) that:
   b) $\Delta f = 0$.
   c) f is constant.

4. Let (M, g) be a closed oriented Riemannian manifold with $Ric \geq 0$. Suppose that $b_1(M) = k$. Show that the Riemannian universal covering $(\tilde{M}, \tilde{g})$ is isometric to $(N, h) \times (\mathbb{R}^k, g_{stand})$ for some Riemannian manifold (N, h).

Please hand in your solutions on Monday, the 14th of June, in the lecture.