Mathematisches Institut
Universität München

Mark Hamilton
Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 5

1. Let $\tilde{M}$ be a complete simply-connected n-dimensional Riemannian manifold of constant sectional curvature $K = 1$. Let p, p' be points on S^n with antipodal points q, q' such that all four points are different. Use suitably defined local isometries $f : S^n \setminus \{q\} \to \tilde{M}$, $f' : S^n \setminus \{q'\} \to \tilde{M}$ to show that $\tilde{M}$ is isometric to S^n.

2. Let M be a differentiable manifold and G a group which acts *properly discontinously* on M, i.e. for each $p \in M$ there exists a neighbourhood $U \subset M$ such that $U \cap g \cdot U = \emptyset$ for all $g \neq e$. Show that the quotient M/G is a manifold, the projection $M \to M/G$ is a covering map and the number of sheets equals $|G|$.

3. a) Let $\Gamma \subset O(2m+1)$ be a subgroup which acts properly discontinously on S^{2m} and let $\gamma \in \Gamma$. Consider the eigenvalues of γ to show that if $det\, \gamma = 1$ then $\gamma = Id$ and if $det\, \gamma = -1$ then $\gamma = -Id$.

b) Let M^{2m} be an even dimensional complete Riemannian manifold with constant sectional curvature $K = 1$. Prove that M is isometric to S^{2m} or $\mathbb{R}P^{2m}$.

4. Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$, p a prime number and q an integer coprime to p. Consider the action

$$\mathbb{Z}_p \times S^3 \to S^3,\ [k] \cdot (z, w) = (e^{2\pi i k/p} z, e^{2\pi i k q/p} w).$$

a) Show that the action is properly discontinous and isometric.

The quotient space $L(p, q) = S^3/\mathbb{Z}_p$ is a manifold and is called a (p, q)-*lens space*. The standard Riemannian metric on S^3 induces on $L(p, q)$ a metric of constant sectional curvature $K = 1$.

b) Prove that two lens spaces $L(p, q)$ and $L(p, q')$ are isometric if and only if $qq' \equiv \pm 1 \bmod p$ or $q \equiv \pm q' \bmod p$. (Use without proof that two manifolds S^3/Γ_1 and S^3/Γ_2 with $\Gamma_1, \Gamma_2 \subset O(4)$ are isometric if and only if Γ_1, Γ_2 are conjugate subgroups.)

Please hand in your solutions on Thursday, the 3rd of June, in the lecture.