Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 4

1. Consider $\mathbb{R}^n$ with the standard Euclidean metric g_0. The group of affine isometries is defined by

$$Iso^{aff}(\mathbb{R}^n, g_0) = \{\phi : \mathbb{R}^n \to \mathbb{R}^n \mid \phi(v) = A \cdot v + t, \text{ for some } A \in O(n), t \in \mathbb{R}^n\}.$$

a) Show that the group of translations is a normal subgroup in $Iso^{aff}(\mathbb{R}^n, g_0)$, with quotient isomorphic to $O(n)$.

b) Prove that every isometry of $(\mathbb{R}^n, g_0)$ is affine, hence $Iso^{aff}(\mathbb{R}^n, g_0) = Iso(\mathbb{R}^n, g_0)$.

2. Consider $\mathbb{R}^{n+1}$ with the quadratic form $Q(v,w) = -v_0w_0 + \sum_{i=1}^n v_iw_i$.

a) Show that $H^n = \{v \in \mathbb{R}^{n+1} \mid Q(v,v) = -1, v_0 > 0\}$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ and the restriction of Q to TH^n induces a Riemannian metric g_Q on H^n.

b) Prove that (H^n, g_Q) is isometric to *hyperbolic n-space* $(\mathbb{H}^n, g)$, where $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ and g is given by

$$g_{ij} = \frac{1}{x_n^2}\delta_{ij}.$$

(*Hint:* You can do this in two steps. First project H^n to the open ball $\mathring{D}^n = \{x = (0, x_1, ..., x_n) \mid |x| < 1\}$ along the lines through $(-1, 0, ..., 0)$ and then map $\mathring{D}^n$ onto the upper half-plane $\mathbb{H}^n$ by inversion at $t = (0, ..., 0, -1)$, i.e. by the map $x \mapsto t + 2\frac{(x-t)}{|x-t|^2}$.)

3. Consider hyperbolic n-space $(\mathbb{H}^n, g)$ as in Ex. 2.

a) Show that $(\mathbb{H}^n, g)$ has constant sectional curvature -1.

b) Prove that the isometry group of $(\mathbb{H}^n, g)$ is isomorphic to $O^+(1,n) = \{\phi \in End(\mathbb{R}^{n+1}) \mid Q(\phi v, \phi w) = Q(v,w),\ \phi(H^n) \subset H^n\}$, with Q as above.

4. a) Let M, N be differentiable manifolds and $\phi : N \to M$ a surjective differentiable map of constant rank. Show that ϕ is a submersion.

b) Let $G \times M \to M$ be a smooth Lie group action. Suppose that M is connected and the action is transitive. Show that the induced action of the identity component G_0 is still transitive.

Please hand in your solutions on **Monday**, the 24th of May, in the lecture.