Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 3

1. Let $G \times M \to M$ be a smooth group action and $p, q \in M$ points on the same orbit. Show that the stabilizers G_p and G_q are conjugate as subgroups of G (and hence isomorphic). Find an example where $G_p \neq G_q$.

2. a) Find an example of a smooth S^1-action whose orbits are not all diffeomorphic.

b) Consider the unitary group $U(n) = \{A \in Mat(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^T = I\}$. Show that the standard linear action of $U(n)$ on $\mathbb{C}^n$ induces a transitive action on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ and determine the stabilizer of a point.

c) Realise the Grassmannian

$$Gr(k,n) = \{V \mid V \subset \mathbb{R}^n \text{ } k\text{-dimensional linear subspace}\},$$

where $0 \leq k \leq n$, as a homogeneous space.

3. Let $\phi : G \times M \to M$ be a smooth (left) action of a Lie group G on a manifold M. The action induces a map

$$\phi_* : \mathfrak{g} \to \mathfrak{X}(M),$$

given by $(\phi_* X)(p) = \frac{d}{dt}|_{t=0}(exp(tX) \cdot p)$. Show that

$$[\phi_* X, \phi_* Y] = -\phi_*[X,Y], \quad \forall X, Y \in \mathfrak{g}.$$

(The minus sign would not be there, if we considered actions from the right.)

4. Let $G \times M \to M$ be a smooth group action. If G is compact, show that every orbit is an embedded submanifold of M.

Please hand in your solutions on Thursday, the 13th of May, in the lecture.