Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 2

1. a) Let M, N be differentiable manifolds and $f : M \to N$ a smooth map of constant rank k. Show that the connected components of the preimages $f^{-1}(y)$, where y runs over $f(M)$, form the leaves of a codimension k foliation on M.

b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the map

$$f(x, y, z) = (1 - x^2 - y^2)e^z.$$

Check that this is a submersion. By a), the preimages form the leaves of a codimension 1 foliation $\mathcal{F}$ on $\mathbb{R}^3$. This foliation is called the *Reeb foliation*. Sketch a picture of $\mathcal{F}$. Denote the preimage of $s \in \mathbb{R}$ by L_s. Prove that $\mathcal{F}$ is invariant under translations in the z-direction and that $L_0 \cong S^1 \times \mathbb{R}$ and $L_s \cong \mathbb{R}^2$ for $s > 0$.

2. Consider the 1-form $\alpha = dx + zdy$ on $\mathbb{R}^3$ with coordinates (x, y, z).

a) Prove that $\ker\alpha = \{X \in T\mathbb{R}^3 \mid i_X\alpha = 0\}$ is a smooth rank 2 distribution $\mathcal{D}$ on $\mathbb{R}^3$.

b) Show that $\mathcal{D}$ is nowhere integrable.

3. Consider the Lie group $SL(2, \mathbb{R})$ with Lie algebra $sl(2, \mathbb{R})$.

a) Show that $sl(2, \mathbb{R}) = \{A \in \mathrm{Mat}(2 \times 2, \mathbb{R}) \mid \mathrm{Tr}A = 0\}$.

b) Prove that the exponential map of $SL(2, \mathbb{R})$ is not surjective. (*Hint*: You can consider $\mathrm{Tr}\exp A$ for $A \in sl(2, \mathbb{R})$.)

4. Let $\phi : H \to G$ be a smooth homomorphism between Lie groups. Prove that $\phi(H)$ is a Lie subgroup of G.

Please hand in your solutions on Thursday, the 6th of May, in the lecture.