Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 11

1. a) Let (M, g) be a Riemannian manifold, γ a geodesic and $J(t)$ a Jacobi field along γ with $J(0) = 0$. Prove that $J(t)$ is orthogonal to $\dot{\gamma}(t)$ for all t if and only if $\dot{J}(0) \perp \dot{\gamma}(0)$.

b) Suppose (M, g) has constant curvature K. Let $w(t)$ be a parallel vector field along the geodesic γ which is orthogonal to $\dot{\gamma}$ and has length 1. Solve in each case $K > 0, K = 0, K < 0$ the Jacobi equation for $J(t)$ with initial conditions $J(0) = 0, \dot{J}(0) = w(0)$.

2. Let (M, g) be a Riemannian manifold, $p \in M$ be a point and $v, w \in T_pM$. Consider a geodesic $\gamma(t)$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$. Show that the vector field J along γ given by $J(t) = (D_{tv}exp_p)(tw)$ solves the Jacobi equation

$$\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0,$$

with initial values $J(0) = 0, \dot{J}(0) = w$.

3. a) Let M be a manifold which admits a complete Riemannian metric with $Ric \geq k > 0$. Show that M does not admit a metric of non-positive sectional curvature.

b) Let M, N be closed manifolds of positive dimension. Show that $M \times N$ does not admit a metric of negative sectional curvature.

4. Let (M, g) be a closed Riemannian manifold of negative sectional curvature. Show that any two commuting elements of the fundamental group $\pi_1(M)$ belong to a common cyclic subgroup and that $\pi_1(M)$ is a uniquely determined union of infinite cyclic subgroups which intersect pairwise only in the identity element.

Please hand in your solutions on Monday, the 19th of July, in the lecture.