Mathematisches Institut
Universität München

Mark Hamilton
Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 10

1. Let $\pi : (\bar{M}, \bar{g}) \to (M, g)$ be a submersion between Riemannian manifolds. The tangent bundle $T\bar{M}$ splits into the vertical and horizontal bundle, $T\bar{M} = V \oplus H$, where $V = T\pi$ is the tangent bundle along the fibres and $H = V^{\perp}$. For a vector field X on M denote by $\bar{X} \in \mathfrak{X}(\bar{M})$ the *horizontal lift*, i.e. the unique horizontal vector field on $\bar{M}$ such that $D\pi(\bar{X}) = X$. The submersion π is called *Riemannian* if $D\pi : H \to TM$ is an isometry on each fibre. Let π be a Riemannian submersion. Show that the Levi-Civita connections $\bar{\nabla}, \nabla$ of $\bar{M}$ and M are related by

$$\bar{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X}, \bar{Y}]^v,$$

for all vector fields X, Y on M, where Z^v denotes the vertical component of $Z \in \mathfrak{X}(\bar{M})$.

2. Let $\pi : (\bar{M}, \bar{g}) \to (M, g)$ be a Riemannian submersion. Denote the sectional curvatures of $\bar{M}, M$ by $\bar{K}$ and K. Prove O'Neill's formula:

$$K(X, Y) = \bar{K}(\bar{X}, \bar{Y}) + \frac{3}{4}|[\bar{X}, \bar{Y}]^v|^2,$$

for all orthonormal vector fields X and Y on M.

3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *bi-invariant metric* is a Riemannian metric on G which is invariant under all left- and right-translations. (One can show that every compact Lie group admits a bi-invariant metric). Suppose G admits a bi-invariant metric g. Denote the Levi-Civita connection of g by ∇. Prove the following formulas for $X, Y, Z, W \in \mathfrak{g}$:

- $g([Y, X], Z) + g(Y, [Z, X]) = 0.$
- $\nabla_X Y = \frac{1}{2}[X, Y].$
- $g(R(X, Y)Z, W) = \frac{1}{4} g([X, Y], [Z, W]).$

Hint: Recall that $[X, Y] = \frac{d}{dt}|_{t=0}(Ad(exp(tX))Y)$, where $Ad(h) = D_e(L_h R_{h^{-1}})$.

4. Let G be a Lie group.

a) Show that if G admits a bi-invariant metric and $\mathfrak{g}$ has trivial center, then G is compact and $\pi_1(G)$ is finite. Prove that $SL(2,\mathbb{R})$ does not admit a bi-invariant metric.

b) Show that if G is compact and non-abelian and M any compact manifold, then $M \times G$ admits a metric of positive scalar curvature.

Please hand in your solutions on Monday, the 12th of July, in the lecture.