Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 SS 04

Geometry of Manifolds II (Prof. Kotschick)
Problem sheet 1

1. Let G be a Lie group. Show that the identity component, i.e. the connected component containing the identity element, is a normal subgroup of G.

2. The *special unitary group* is defined as

$$SU(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A \cdot \bar{A}^T = I, \quad \det A = 1\}.$$

Show that $SU(n)$ is a compact Lie group of dimension $n^2 - 1$ and determine its Lie algebra $su(n)$. Prove that, as a manifold, $SU(2)$ is diffeomorphic to S^3.

3. a) Let G be a Lie group and $X, Y \in \mathfrak{g}$. Show that the curve

$$exp(tX)exp(tY), \quad t \in \mathbb{R}$$

has initial velocity vector $X + Y$.

b) Find an explicit example where $exp(X)exp(Y) \neq exp(X + Y)$.

4. Consider the general linear group $GL(n, \mathbb{R})$ with Lie algebra $gl(n, \mathbb{R})$. Show that as a vector space, $gl(n, \mathbb{R})$ can be canonically identified with $\mathrm{Mat}(n \times n, \mathbb{R})$. Prove that the (abstract) exponential mapping is given by

$$exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad \forall X \in gl(n, \mathbb{R}),$$

and that

$$[X, Y] = XY - YX \quad \forall X, Y \in gl(n, \mathbb{R}).$$

Please hand in your solutions on Thursday, the 29th of April, in the lecture.