Mathematisches Institut
Universität München

Mark Hamilton
Office hour: Mon 1-2 pm, Room 308 WS 03/04

## Geometry of Manifolds I (Prof. Kotschick)
## Problem sheet 2

1. Let $F$ be a differentiable manifold and $\phi : F \to F$ a diffeomorphism. Consider the topological space $F \times [0,1]$ with the equivalence relation $\sim$ given by

$$\begin{aligned}(x,t) \sim (y,s) \quad \Leftrightarrow \quad & (x,t) = (y,s) \\ & \text{or} \quad t = 0, s = 1 \quad \text{and} \quad y = \phi(x) \\ & \text{or} \quad s = 0, t = 1 \quad \text{and} \quad x = \phi(y).\end{aligned}$$

Let $M = (F \times [0,1])/_\sim$ denote the quotient space. This is called the *mapping torus* of $\phi$.

a) Show that $\sim$ is an equivalence relation. The quotient is clearly Hausdorff and paracompact. Define explicitly a differentiable structure on $M$, induced from the structure on $F$.

Important examples of mapping tori are the 2-torus $T^2$ ($F = S^1, \phi(x) = x$), the Klein bottle $K^2$ ($F = S^1, \phi(x) = -x$), the Möbius band ($F = \mathbb{R}, \phi(x) = -x$) and the twisted 2-sphere bundle over the circle, $S^2 \tilde{\times} S^1$ ($F = S^2, \phi(x) = -x$).

b) Show that a line bundle (i.e. a vector bundle of rank 1) over a manifold $X$ is trivial if and only if it admits a nowhere vanishing section. Show that the tangent bundle of $S^1$ is trivial. Show that the Möbius band is a line bundle over the circle $S^1$ and prove that it is non-trivial.

2. Consider the equivalence relation $\sim$ on $\mathbb{R}^n$ given by

$$x \sim y \Leftrightarrow \exists k \in \mathbb{Z}^n : y = x + k.$$

The quotient is denoted by $\mathbb{R}^n/\mathbb{Z}^n$ and is a smooth manifold, diffeomorphic to the n-torus $T^n = S^1 \times ... \times S^1$ (you do not have to prove this).

Let $SL_2(\mathbb{Z})$ be the group of $2 \times 2$-matrices with integer entries and determinant 1. Let $A \in SL_2(\mathbb{Z})$ act on $\mathbb{R}^2$ by matrix multiplication (the elements of $\mathbb{R}^2$ are considered as column vectors). This is a diffeomorphism. Show that it descends to a diffeomorphism

$$\phi_A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2.$$

Let $M_A$ denote the mapping torus of $\phi_A$. This is a 3-manifold which is in general not diffeomorphic to $T^3$. Show that $M_A$ has trivial tangent bundle.

3. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus 0$ by $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus 0 : x = \lambda y$. Real projective n-space is the space of equivalence classes

$$\mathbb{R}P^n := (\mathbb{R}^{n+1} \setminus 0)/_\sim$$

with the quotient topology. The equivalence class of a point $(x_0, ..., x_n)$ is denoted by $[x_0 : ... : x_n]$.

Show that $\mathbb{R}P^n$ is a compact n-dimensional differentiable manifold (you can use the sets $U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0\}$ for $i = 0, ..., n$ as charts). Let

$$\pi : S^n \to \mathbb{R}P^n, (x_0, ..., x_n) \mapsto [x_0 : ... : x_n]$$

be the restriction of the canonical projection. Show that $\pi$ is surjective and that the preimage of every $p \in \mathbb{R}P^n$ consists of two points (i.e., $S^n$ is a two-sheeted covering of $\mathbb{R}P^n$).

4. Similarly one defines complex projective n-space as

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus 0)/_\sim, \quad \text{where} \quad z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C} \setminus 0 : z = \lambda w.$$

As above, one can show that this is a compact 2n-dimensional differentiable manifold. Consider the restriction of the canonical projection to $S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Show that this is a submersion. What is the preimage of a point in $\mathbb{C}P^n$?

Please hand in your solutions on Thursday, the 6th of November, in the lecture.