Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 14

1. a) Show that an n-dimensional Riemannian manifold is Einstein with Einstein constant d if and only if $Q(X,Y) = d(n-1)\langle X, Y\rangle$ for all vectors X, Y.

b) Let $M \times N$ be the product of Riemannian manifolds with the product metric. Show that for $X \in \pi_1^* TM$ and $Y \in \pi_2^* TN$, $K(X,Y) = 0$.

2. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ with the induced metric g. Consider the product $M = S^k \times S^l$ with the metric $\langle\,,\rangle = \lambda_1 g_1 \oplus \lambda_2 g_2$ where $\lambda_1, \lambda_2 > 0$. For which values of the parameters λ_1, λ_2 is $(M, \langle\,,\rangle)$ Einstein?

3. Let $(M, \langle\,,\rangle)$ be a Riemannian manifold. Use the proof of the fact that the sectional curvature determines the curvature tensor to show that M has constant sectional curvature equal to c if and only if

$$\langle R(X,Y)Z, W\rangle = c \cdot (\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle),$$

for all vectors X, Y, Z, W.

4. Let $\mathbb{H}^2$ be the hyperbolic plane as defined in Ex. 3 sheet 11. The Levi-Civita connection can be computed as

$$\nabla_{\partial_x}\partial_x = (1/y)\partial_y = -\nabla_{\partial_y}\partial_y \quad \nabla_{\partial_x}\partial_y = (1/y)\partial_x = \nabla_{\partial_y}\partial_x.$$

Identify $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$.

a) Show that for fixed $x_0 \in \mathbb{R}$ the curve $s_{x_0}(t) = x_0 + ie^t, t \in \mathbb{R}$, is a geodesic.

b) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), z \in \mathbb{H}^2$ let

$$\phi_A(z) = \frac{az+b}{cz+d}.$$

Show that this defines an isometry $\phi_A : \mathbb{H}^2 \to \mathbb{H}^2$ with $\phi_A^{-1} = \phi_{A^{-1}}$.

c) Prove that for every $z \in \mathbb{H}^2$ there is an $A \in SL(2,\mathbb{R})$ such that $\phi_A(i) = z$. Consider the group $G = \{A \in SL(2,\mathbb{R}) \mid \phi_A(i) = i\}$. Prove that for every $v \in T_i\mathbb{H}^2$ with $||v|| = 1$ there is an $A \in G$ such that $D_i\phi_A(\partial_y) = v$.

d) Show that every geodesic of $\mathbb{H}^2$ is of the form $\phi_A \circ s_0(rt)$ for some $A \in SL(2,\mathbb{R})$ and some $r > 0$.

Please hand in your solutions on Thursday, the 12th of February, in the lecture.