Mathematisches Institut
Universität München

Mark Hamilton
Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 13

1. a) Let $\Delta \subset \mathbb{R}^2$ be a triangle. Denote the vertices by A_i, the center of the edges by B_i and the center of the triangle by C. Sketch a vector field on Δ with singularties A_i, B_i, C, where A_i and C have index 1 and the B_i have index -1.

b) Show that for every triangulation of a closed surface Σ:

$$\#\,\text{vertices} - \#\,\text{edges} + \#\,\text{faces} = e(\Sigma),$$

where $e(\Sigma)$ denotes the Euler number.

2. Let $(N, \langle\,,\rangle)$ be a Riemannian manifold and $M \subset N$ a submanifold with the induced metric. Denote by $\mathfrak{X}(M)^\perp$ the vector fields along M which are normal to M. For $X, Y \in \mathfrak{X}(M)$ and $p \in M$ define

$$B(X,Y)(p) = (\nabla^N_{\tilde{X}}\tilde{Y} - \nabla^M_X Y)(p),$$

where $\tilde{X}, \tilde{Y}$ are local extensions of X, Y in a neighbourhood of p.

a) Show that this expression does not depend on the choice of extensions $\tilde{X}, \tilde{Y}$ and gives a well defined map $B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$.

b) Show that the map B is linear over $C^\infty(M)$ in both arguments and is symmetric.

3. Let $(N, \langle\,,\rangle)$ be a Riemannian manifold and $M \subset N$ a submanifold with the induced metric.

a) Let $W \in \mathfrak{X}(M)^\perp$ and $\tilde{W}$ a local extension to an open subset of N. Show that for all $X, Y \in \mathfrak{X}(M)$:

$$\langle \nabla^N_X \tilde{W}, Y \rangle = -\langle B(X,Y), \tilde{W} \rangle.$$

b) Let X, Y, Z, T be vector fields tangent to M. Prove the following generalized Gauss equation:

$$\begin{aligned}\langle F^{\nabla^N}(X,Y)Z, T\rangle &= \langle F^{\nabla^M}(X,Y)Z, T\rangle \\ &\quad -\langle B(X,T), B(Y,Z)\rangle + \langle B(Y,T), B(X,Z)\rangle.\end{aligned}$$

4. a) Let N, M be as above. Consider the vector bundle $TM^\perp \to M$ whose fibre over $p \in M$ is the orthogonal complement of T_pM in T_pN. The metric on TN induces a metric on $TM^\perp$. Let τ denote the orthogonal projection $TN|_M \to TM^\perp$. Show that for $X \in \mathfrak{X}(M), W \in \mathfrak{X}(M)^\perp$, with local extensions $\tilde{X}, \tilde{W}$,

$$\nabla^\perp_X W = \tau(\nabla^N_{\tilde{X}} \tilde{W})$$

gives a well-defined connection $\nabla^\perp$ on $TM^\perp$ which is compatible with the metric.

b) For $X, Y, Z \in \mathfrak{X}(M), W \in \mathfrak{X}(M)^\perp$ let $\bar{B}(X, Y, W) = \langle B(X, Y), W\rangle$. The covariant derivative $\nabla \bar{B}$ of this tensor is defined by (compare with Ex. 3 sheet 10)

$$\begin{aligned}(\nabla_X \bar{B})(Y, Z, W) &= L_X(\bar{B}(Y, Z, W)) - \bar{B}(\nabla^M_X Y, Z, W) \\ &\quad -\bar{B}(Y, \nabla^M_X Z, W) - \bar{B}(Y, Z, \nabla^\perp_X W).\end{aligned}$$

Prove the following generalized Codazzi equation:

$$\langle F^{\nabla^N}(X, Y)Z, W\rangle = (\nabla_X \bar{B})(Y, Z, W) - (\nabla_Y \bar{B})(X, Z, W).$$

Please hand in your solutions on Thursday, the 5th of February, in the lecture.