Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 11

1. Let $E \to M$ be an oriented rank 2 vector bundle. Prove that if E admits a nowhere vanishing smooth section then the Euler class $e(E)$ vanishes. Show that every smooth vector field on S^2 has a zero.

2. a) Let $(M, \langle\,,\rangle)$ be a Riemannian manifold with Levi-Civita connection ∇. Show that in a local chart the Christoffel symbols of ∇ are given by

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}),$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) with entries $g_{ij} = \langle \partial_i, \partial_j \rangle$.

b) Define the components R_{ijk}^s of the curvature tensor in a local chart by $R(\partial_i, \partial_j)\partial_k = \sum_s R_{ijk}^s \partial_s$. Show that they are given in terms of the Christoffel symbols by

$$R_{ijk}^s = \sum_l (\Gamma_{jk}^l \Gamma_{il}^s - \Gamma_{ik}^l \Gamma_{jl}^s) + \partial_i \Gamma_{jk}^s - \partial_j \Gamma_{ik}^s.$$

3. The hyperbolic plane is defined as $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric $\langle\,,\rangle$ which is given in the standard basis (∂_x, ∂_y) by

$$g_{ij}(x,y) = \frac{1}{y^2}\delta_{ij}.$$

Let ∇ denote the Levi-Civita connection of $\langle\,,\rangle$. Compute in a global orthonormal frame the matrix of connection 1-forms ω and the matrix of curvature 2-forms Ω. Show that $\Omega_{12} = dvol_g$, i.e. the Gauss curvature of $\langle\,,\rangle$ is of constant value -1.

4. Let M be a differentiable manifold. Suppose ∇ is an affine connection with torsion T and curvature R. Define the covariant derivatives ∇T and ∇R by

$$\begin{aligned}(\nabla_X T)(Y,Z) &= \nabla_X(T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z)\\ (\nabla_X R)(Y,Z)W &= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W\\ &\quad - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W.\end{aligned}$$

a) Use the fact that T and R are linear over $C^\infty(M)$ in all arguments to show that the same holds for ∇T and ∇R.

b) Prove the following identities:

$$\begin{aligned}
\sum(\nabla_X T)(Y,Z) &= \sum R(X,Y)Z - \sum T(T(X,Y),Z) \\
\sum(\nabla_X R)(Y,Z)W &= -\sum R(T(X,Y),Z)W.
\end{aligned}$$

Here $\sum$ denotes summation over the cyclic permutations of X, Y, Z.

Please hand in your solutions on Thursday, the 22nd of January, in the lecture.