Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 10

1. Let $E \to M$ be a vector bundle with connection ∇. Recall that the curvature $F^\nabla \in \Omega^2(End\, E)$ is defined via the composition $\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E)$. Show that

$$F^\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for all vector fields X, Y and all sections s of E.

2. Let $\pi : E \to M$ be an oriented rank 2 vector bundle over a manifold M and suppose $\langle\, ,\, \rangle_0$ and $\langle\, ,\, \rangle_1$ are two metrics on E. Then the vector bundle

$$E \times [0,1] \to M \times [0,1], (v,t) \mapsto (\pi(v), t),$$

has a metric $\langle\, ,\, \rangle$ which on $E \times \{t\} \to M \times \{t\}$ is given by

$$\langle\, ,\, \rangle_t = (1-t)\langle\, ,\, \rangle_0 + t\langle\, ,\, \rangle_1.$$

Use this construction and Ex. 2 on sheet 6 to show that the Euler class $e(E, \langle\, ,\rangle) \in H^2_{DR}(M)$ of an oriented rank 2 vector bundle is independent of the chosen metric.

3. a) Let M be a smooth manifold of dimension > 0. Show that there is no affine connection ∇ on M such that $\nabla_X Y = \nabla_Y X$ for all vector fields X, Y.

b) Let ∇ be an affine connection. Define the covariant derivative of $\omega \in \Omega^k(M)$ by

$$(\nabla^*_X \omega)(Y_1, ..., Y_k) = L_X(\omega(Y_1, ..., Y_k)) - \sum_{i=1}^{k} \omega(Y_1, ..., \nabla_X Y_i, ..., Y_k).$$

$\nabla^*_X \omega$ is again a k-form. Suppose ∇ is torsion-free. Show for $\omega \in \Omega^2(M)$: if $\nabla^* \omega = 0$ then $d\omega = 0$.

4. Let (M, g) be a Riemannian manifold and $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ a $C^\infty(M)$-bilinear, antisymmetric mapping. Show that there exists a unique affine Riemannian connection on M which has torsion T.

Please hand in your solutions on Thursday, the 15th of January, in the lecture.