Geometry of manifolds I
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(1) Topological spaces; open, closed and compact sets; bases for the topology; Hausdorffness; continuity.
(2) Metric spaces, $\mathbb{R}^n$.
(3) **Topological manifolds** are locally Euclidean spaces that are Hausdorff and second countable; charts and atlases.
(4) Every manifold M is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering of M, there is an atlas $\{(V_k, \varphi_k)\}$ such that the covering by the V_k is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball B_3 of radius 3 for all k and the open sets $W_k = \varphi_k^{-1}(B_1)$ cover M.

Proof. We prove first that there is a sequence G_i, $i = 1, 2, \ldots$ of open sets with compact closures, such that the G_i cover M and satisfy
$$\overline{G_i} \subset G_{i+1}$$
for all i. To this end let U_i, $i = 1, 2, \ldots$ be a countable basis of the topology consisting of open sets with compact closures. (This exists because M is second countable, Hausdorff and locally compact.) Set $G_1 = U_1$. Suppose inductively that we have defined
$$G_k = U_1 \cup \ldots \cup U_{j_k} \ .$$
Then let j_{k+1} be the smallest integer greater than j_k with the property that
$$\overline{G_k} \subset U_1 \cup \ldots \cup U_{j_{k+1}} \ ,$$
and define
$$G_{k+1} = U_1 \cup \ldots \cup U_{j_{k+1}} \ .$$
This defines the sequence G_k as desired.

Now let $\{U_\alpha \mid \alpha \in A\}$ be an arbitrary open covering of M. For every $x \in M$ we can find a chart (V_x, φ_x) at x with V_x contained in one of the U_α and such that $\varphi_x(V_x) = B_3$. Let $W_x = \varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_i} \setminus G_{i-1}$ by finitely many such W_{x_j} such that at the same time the corresponding V_{x_j} are contained in the open set $G_{i+1} \setminus \overline{G_{i-2}}$. Taking all these

V_{x_j} as i ranges over the positive integers we obtain the desired atlas. $\square$

(5) **Smooth manifolds** as topological manifolds with an atlas whose transition maps are diffeomorphisms of some class C^k, $k \geq 1$. Examples: Euclidean spaces, spheres, tori, $GL(n, \mathbb{R})$,...
(6) Differentiable functions and maps; diffeomorphisms vs. homeomorphisms.

18 October 2001

(7) Unions of atlases; equivalence and maximality.
(8) Dimensions of manifolds and smooth invariance of domain.
(9) Every open covering of a smooth manifold admits a subordinate smooth **partition of unity**. This follows from paracompactness and the existence of smooth cutoff functions.
(10) The **structure cocycle** of an atlas. Cocycles with values in arbitrary pseudogroups of homeomorphisms between open subsets of $\mathbb{R}^n$, and the corresponding classes of manifolds. Compare Sections 3.1 of [1] and of [3].

A **pseudogroup** (on $\mathbb{R}^n$) is a non-empty set $\mathcal{G}$ of homeomorphisms $h\colon D(h) \to R(h)$ between open sets (of $\mathbb{R}^n$) satisfying the following conditions:
(a) If $g, h \in \mathcal{G}$ and $R(h) \subset D(g)$, then $g \circ h \in \mathcal{G}$.
(b) If $h \in \mathcal{G}$, then $h^{-1} \in \mathcal{G}$.
(c) The restriction of any $h \in \mathcal{G}$ to an open subset contained in $D(h)$ is also in $\mathcal{G}$.
(d) If $h\colon U \to V$ is a homeomorphism between open sets with the property that U can be covered by open sets U_i such that $h|_{U_i} \in \mathcal{G}$ for all i, then $h \in \mathcal{G}$.

22 October 2001

(11) Inclusions between pseudogroups and subordination of $\mathcal{G}$-structures.
(12) Examples of pseudogroups and the corresponding classes of manifolds: C^k-manifolds for $k \in \{0, 1, 2, \ldots, \infty, \omega\}$, complex manifolds, oriented manifolds, measured manifolds, foliated manifolds.

25 October 2001

(13) Pseudogroups on $\mathbb{H}^n$ and **manifolds with boundary**.
(14) **Vector bundles**; the **tangent bundle** of a smooth manifold, see [1] Section 3.3.

29 October 2001

(15) **Derivatives of smooth maps** between manifolds; tangent vectors as derivatives of paths.
(16) Existence of metrics on vector bundles; **Riemannian metrics** and length of tangent vectors.

5 November 2001

(17) **Arclength** and **distance** in a Riemannian manifold; the induced metric space structure.
(18) A subset N of an m-dimensional manifold M is a **submanifold** of dimension n if every point in N has an open neighbourhood U in M which is the domain of a chart (U, ϕ) such that

$$\phi(U \cap N) = \phi(U) \cap \mathbb{R}^n \subset \mathbb{R}^m \ .$$

This is called a "properly embedded submanifold" in [1].
Immersions and **embeddings** (=injective immersions). Every (compact) manifold can be embedded in a Euclidean space.

8 November 2001

(19) The image of any proper embedding is a submanifold. Examples of non-proper embeddings.
(20) **Regular** and **critical values** of smooth maps, **submersions**.
(21) **Cocycles** and G-**structures** on vector bundles, see [1] Section 3.4. Examples for G: the trivial group, the group $GL(l, k-l) \subset Gl(k, \mathbb{R})$ preserving a subspace of dimension l.

12 November 2001

(22) Further examples: the identity component of $Gl(k, \mathbb{R})$, the orthogonal group, the complex linear group.
(23) **Infinitesimal geometric structures** on manifolds; the **integrability problem**.

15 November 2001

(24) Integration of **vector fields**; **local** and **global flows**; **completeness**, see [1] Section 4.1.
(25) Vector fields as **derivations** and the **Lie derivative**.

19 November 2001

(26) All derivations are vector fields.
(27) **Commutators** and the Lie algebra structure on the space of vector fields, see [1] Section 4.2.

22 November 2001

(28) Commuting vector fields and k-flows.

26 November 2001

(29) More on k-flows and the integrability condition for (infinitesimal) parallelizations, see [1] Section 4.3.

29 November 2001

(30) The integrability problem for infinitesimal $GL(l, n-l)$-structures, see [1] Section 4.4. Infinitesimal $GL(l, n-l)$-structures are distributions, and in this case the integrability criterion is given by the **Frobenius theorem**.

3 December 2001

(31) **Lie groups** and their **Lie algebras**: definitions, examples.

6 December 2001

(32) Completeness of left-invariant vector fields, and their **one-parameter subgroups**. The **exponential map**.

13 December 2001

(33) **Lie subgroups**. The correspondence between connected Lie subgroups and Lie subalgebras.

17 December 2001

(34) Actions of Lie groups on manifolds; **homogeneous spaces**.

20 December 2001

(35) The Lie groups $SO(3)$, S^3 and $SU(2)$; and their Lie algebra.

7 January 2002

(36) The cotangent bundle; **one-forms** and their functorial properties, see [1] Section 6.1.

10 January 2002

(37) **Integrals** of one-forms; exactness and local exactness. (See [1] Section 6.2.)

14 January 2002

(38) The first **de Rham cohomology** group: definition and homotopy invariance. (See [1] Section 6.3.)

17 January 2002

(39) The calculation of $H^1(S^1)$, and some topological applications, including the **Brouwer fixed point theorem** in the plane. (See [1] Section 6.3.)
(40) The **tensor algebra** of a real vector space.

21 January 2002

(41) **Exterior algebra**. (See [1] Section 7.2.)

24 January 2002

(42) **Differential forms** and the exterior derivative.

28 January 2002

(43) **Closed** and **exact** p-froms; the **de Rham cohomology algebra**.
(44) Existence and uniqueness of the **integral** of n-forms with compact support on oriented n-manifolds. (See [1] Section 8.2.)

31 January 2002

(45) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \ .$$

(See [1] Section 8.2.)

4 February 2002

(46) The **Poincaré Lemma**. It implies the homotopy-invariance of de Rham cohomology. (See [1] Section 8.3.)

(47) The de Rham cohomology of S^n. Application to the **Brouwer fixed point theorem** in dimension $n + 1$.

7 February 2002

(48) Contractions and **Lie derivatives** of differential forms; the Cartan formula $L_X = i_X d + d i_X$. **Divergence** of a vector field. (See [2] Section 2.24.)

(49) The **Hopf invariant** for maps $S^{2n-1} \to S^n$, and its generalisation to divergence-free vector fields on S^3.

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.
2. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag 1983.
3. W. P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1, Princeton University Press 1997.