CHAPTER 4

Non–existence results

1. The intersection form

The pairwise intersections of embedded oriented surfaces in a closed oriented 4–manifold X define a symmetric bilinear form Q_X on its second homology, called the *intersection form* of X, which is dual to the cup product in cohomology. It is a consequence of Poincaré duality that the intersection form is unimodular and in particular non–degenerate on the lattice $H_2(X,\mathbb{Z})/\mathrm{Torsion}$.

EXAMPLE 4.1. Consider $X = \mathbb{C}P^2$. Its homology is generated by the class of a complex line, whose self–intersection number is 1 because two different lines intersect in one (positive) point. Thus $Q_{\mathbb{C}P^2} = (1)$. If we reverse the orientation, i.e. we consider $\mathbb{C}P^2$ with the orientation opposite to the one induced by the complex structure, then the intersection number changes sign.

More generally, if X is oriented and $\overline{X}$ denotes the same manifold with the orientation reversed, then $Q_{\overline{X}} = -Q_X$.

By the work of Whitehead–Milnor and Freedman respectively, the intersection form determines the homotopy type and even the homeomorphism type of a simply connected smooth 4–manifold. However, as discovered by Donaldson, Q_X does not determine the diffeomeorphism type, and we shall exhibit examples to this effect in later chapters.

Freedman also proved that every unimodular form occurs as the intersection form of a simply connected topological 4–manifold. That this is not true for differentiable manifolds follows already from Rokhlin's theorem 1.15, and further restrictions were found by Donaldson. In this Chapter we prove the known restrictions imposed on Q_X by the requirement that X be smooth. These restrictions have to do with the integral structure underlying Q_X.

If we were to take homology with coefficients in a field of characteristic zero, such as $\mathbb{R}$ or $\mathbb{Q}$, then the intersection form would be determined, up to isomorphism over the ground field, by its rank $rk(Q_X) = rk(H_2(X)) = b_2(X)$ and signature $\sigma(Q_X) = \sigma(X) = b_2^+(X) - b_2^-(X)$. Clearly all possible ranks and signatures are realised by the simply connected smooth 4–manifolds $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ as we vary the natural numbers p and q.

Over the integers, forms are not determined by their rank and signature. For example, the hyperbolic form

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not isomorphic over $\mathbb{Z}$ to the diagonal form

$$(1) \oplus (-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

although they have the same rank and signature. These forms are distinguished by their parity. The form H is *even*, meaning that every vector has even selfintersection number, and $(1) \oplus (-1)$ is not. Forms which are not even are called *odd*.

The parity of the intersection form is closely related to the spin condition on a manifold:

LEMMA 4.2. *If X is spin, then Q_X is even. The converse holds if $H_1(X,\mathbb{Z})$ is free of 2–torsion.*

PROOF. Every class in $H_2(X,\mathbb{Z})$ can be represented by a smoothly embedded closed oriented surface $\Sigma \subset X$. We have

$$TX|_{\Sigma} = T\Sigma \oplus \nu ,$$

where ν is the normal bundle of Σ in X.

Recall from Chapter 1 that X is spin if its second Stiefel–Whitney class $w_2(X)$ is zero. As X and Σ are oriented, the Whitney sum formula for w_2 gives

$$\langle w_2(X), [\Sigma]\rangle = \langle w_2(\Sigma), [\Sigma]\rangle + \langle w_2(\nu), [\Sigma]\rangle .$$

On the right–hand side the first summand is zero because Σ is spin, and the second summand is just the mod 2 degree of ν, i.e. the mod 2 reduction of the selfintersection number of Σ.

Thus we see that Q_X is even if and only if $\langle w_2(X), [\Sigma]\rangle$ vanishes for all Σ. This proves the first claim.

Conversely, assume that Q_X is even, so that $w_2(X)\colon H_2(X,\mathbb{Z}) \to \mathbb{Z}_2$ is the zero homomorphism. As the torsion in $H_2(X,\mathbb{Z})$ and in $H^2(X,\mathbb{Z})$ is the same as that in $H_1(X,\mathbb{Z})$ by Poincaré duality and the universal coefficient theorem, the assumption about absence of 2–torsion in $H_1(X,\mathbb{Z})$ implies that $w_2(X)$ vanishes. $\square$

The lemma is sharp, as shown by the following:

EXAMPLE 4.3. Let $X \subset \mathbb{C}P^3$ be a smooth quartic surface defined over $\mathbb{R}$ which has no real points, say $X = \{(z_0, \dots , z_3)|\Sigma z_i^4 = 0\}$. Then complex conjugation acts freely on X and the quotient Y is not spin but has even intersection form $Q_Y = H \oplus E_8$, where E_8 is the negative definite even form associated with the E_8 Dynkin diagramme.

The manifold Y is a complex surface, called the *Enriques surface*, although this is not obvious from our description. One can check the assertion that it is not spin from the construction, or one can invoke Rokhlin's theorem 1.15.

The form E_8 is but the simplest one of the non–diagonalisable definite forms, meaning that it is not equivalent over the integers to the standard form $(-1) \oplus \dots \oplus (-1)$. Other examples are $E_8 \oplus (-1)$ which has rank 9 and is odd, or higher–rank even forms like $E_8 \oplus E_8$ or the Leech lattice which is even of rank 24. The classification of definite unimodular forms over $\mathbb{Z}$ is not understood.

Notice that by Lemma 4.2 and Rokhlin's theorem 1.15, E_8 or the Leech lattice cannot be the intersection forms of simply connected smooth 4–manifolds, though they could be the intersection forms of manifolds with 2–torsion in $H_1(X, \mathbb{Z})$. Of course, Rokhlin's theorem says nothing about $E_8 \oplus E_8$. All these forms are ruled out by the following result which says that among definite forms only the standard ones arise from smooth manifolds, without any assumption on the first homology or the fundamental group.

THEOREM 4.4. (Donaldson) *If a smooth closed oriented 4–manifold has a definite intersection form, then this form diagonalizes over $\mathbb{Z}$.*

We shall prove this in Section 2 below.

In the indefinite case, the classification of unimodular symmetric bilinear forms is known. By the so–called Hasse–Minkowski theorem they are diagonal when odd, and isomorphic to $aH \oplus bE_8$ when even, where $a, b \in \mathbb{Z}$ and $a \geq 1$. We have already seen that the diagonal forms are realised by connected sums of $\mathbb{C}P^2$, so that we are left with the

Question: Which even indefinite forms are realised by smooth 4–manifolds?

We consider the spin case first. By Rokhlin's theorem if $aH \oplus bE_8$ is the intersection form of a spin manifold, then b must be even. One can realise H by $S^2 \times S^2$, and $3H \oplus 2E_8$ by the $K3$ surface, the quartic surface in $\mathbb{C}P^3$ mentioned in the Example 4.3. Both these manifolds are simply connected and spin. Forming connected sums and changing orientations, one can ralise $mH \oplus 2nE_8$ for all $m \geq 3|n|$.

The follwing conjecture says that no other forms should be realised by spin manifolds:

$\frac{11}{8}$**–Conjecture:** If X is a smooth closed oriented spin 4–manifold, then $b_2(X) \geq \frac{11}{8}|\sigma(X)|$.

If one relaxes the spin condition and only requires that the intersection form be even, then more forms are possible as shown by the existence of the Enriques surface. Forming connected sums of $S^2 \times S^2$ and the Enriques surface, one can realise $aH \oplus bE_8$ for all $a \geq |b|$ as intersection forms of smooth 4–manifolds. Again, it is possible that no other forms occur:

Conjecture: If X is a smooth closed oriented 4–manifold with an even intersection form, then $b_2(X) \geq \frac{10}{8}|\sigma(X)|$.

Take this up in Ch. 9...

Both conjectures are satisfied by all compact complex surfaces. Using Theorem 4.5 below and the Hitchin–Thorpe inequality, cf. Chapter **??**, one sees that the conjectures hold for Einstein 4–manifolds, or for Riemannian manifolds satisfying certain pinching conditions on their sectional curvatures. Thus connected sums of complex surfaces and Einstein manifolds also satisfy the conjectures.

In this Chapter we shall prove the following results in the direction of the $\frac{11}{8}$–Conjecture beyond Theorem 4.4:

THEOREM 4.5. *Let X be a smooth closed oriented spin 4–manifold. If its intersection form is $mH \oplus 2nE_8$ with $n \neq 0$, then $m \geq 3$.*

In other words, the $\frac{11}{8}$–Conjecture holds for manifolds with $b_2^+ \leq 2$. This was proved by Donaldson using Yang–Mills theory, under the additional assumption that $H_1(X, \mathbb{Z})$ is free of 2–torsion. In Section 3 we shall give a proof using the monopole moduli spaces.

In Section 4 we shall prove:

THEOREM 4.6. (Furuta) *Let X be a smooth closed orientable spin 4–manifold. If its intersection form is $mH \oplus 2nE_8$, then $m \geq 2n$.*

Equivalently,

$$b_2(X) \geq \frac{10}{8}\sigma(X) \ .$$

1.1. A few words on the fundamental group. By doing surgery on connected sums of copies of $S^1 \times S^3$ one can realise every finitely presentable group as the fundamental group of a smooth closed oriented spin 4–manifold.

We have seen that the absence of 2–torsion in the Abelianised fundamental group $H_1(X, \mathbb{Z})$ can constrain the intersection form of X. However, the free part of $H_1(X, \mathbb{Z})$ has no such effect:

LEMMA 4.7. *If a bilinear form occurs as the intersection form of some closed smooth 4–manifold X, then it also occurs as the intersection form of a 4–manifold Y with prescribed first Betti number. In particular, one can require $b_1(Y) = 0$ and $Q_Y = Q_X$.*

If X is spin, then Y can be taken to be spin as well.

PROOF. It is clear that we can always increase b_1 by forming the connected sum of X with copies of $S^1 \times S^3$. This preserves the intersection form and the property of being spin.

We can also decrease b_1 if it is positive. Let $S^1 \subset X$ be an embedded circle which is a basis element of the free part of $H_1(X, \mathbb{Z})$. By doing elementary surgery on S^1 we decrease b_1 without changing $H_2(X, \mathbb{Z})$ or $w_2(X)$. □

We invite the reader to check that the argument breaks down if the circle has finite order in $H_1(X, \mathbb{Z})$.

Although Lemma 4.7 shows that the first Betti number by itself does not constrain the intersection form, one should keep in mind that the group cohomology of the fundamental group does impose constraints. Namely, if

$$c\colon X \longrightarrow K(\pi_1(X), 1)$$

is the classifying map of the universal covering $\tilde{X} \to X$, then

$$c^*\colon H^*(\pi_1(X)) \longrightarrow H^*(X)$$

is an algebra homomorphism which is an isomorphism in degrees 0 and 1 and is injective in degree 2. Thus, $b_2(X) = rk(Q_X)$ is bounded below by $b_2(\pi_1(X))$. Moreover, the signature is constrained as well, as in the following example:

EXAMPLE 4.8. Let X be a closed oriented 4–manifold with $\pi_1(X) = \mathbb{Z}^2$. Then $c\colon X \to T^2$, and as c^* preserves the cup product it must map $H^2(T^2)$ to an isotropic subspace in $H^2(X)$. Thus Q_X cannot be definite.

In fact, it is not hard to see that for no $k > 1$ can $\mathbb{Z}^k$ be the fundamental group of a manifold with definite intersection form, although $\mathbb{Z}$ is the fundamental group of $S^1 \times S^3$.

2. Proof of Theorem 4.4

Let X be a smooth 4–manifold with definite intersection form. After changing the orientation if necessary, we may assume $b_2^+ = 0$. By Lemma 4.7 we may also assume $b_1 = 0$.

Fixing a Riemannian metric and a Spin^c–structure we consider the moduli space M_ω of solutions to the monopole equations with perturbation parameter $\omega \in i\Omega^2_+$. For each ω there is a reducible solution $(A_\omega, 0)$, because all harmonic forms are ASD as $b_2^+ = 0$. This reducible solution is unique up to gauge equivalence because $b_1 = 0$.

The dimension of the moduli space in this case is

$$\dim M_\omega = \frac{1}{4}(c_1^2(V_\pm) - (2\chi(X) + 3\sigma(X)) = \frac{1}{4}(c_1^2(V_\pm) + b_2(X)) - 1 \ .$$

Note that because $c_1(V_\pm)$ is a characteristic vector for the intersection form, $c_1^2(V_\pm) = \sigma(X) + 8k = -b_2(X) + 8k$, so $\dim M_\omega = 2k - 1$ is an odd number. We now make the following:

ASSUMPTION 4.9. $\dim M_\omega > 0$, equivalently $k > 0$.

Then for generic ω we have a compact non–empty moduli space which is smooth away from the unique reducible solution. We need to understand the neighbourhood of this reducible solution.

The linearised equation at the reducible uncouples into the half de Rham complex (1.2) and the Dirac operator. As $b_1 = b_2^+ = 0$, the half de Rham complex does not contribute to $H^1_{(A_\omega,0)}$ and $H^2_{(A_\omega,0)}$. The complex index of the Dirac operator $D^+_{A_\omega}$ is k, so if its cokernel is $\mathbb{C}^r$ for some $r \geq 0$, its kernel is $\mathbb{C}^{k+r}$. Thus, the Kuranishi model of a neighbourhood of $[A_\omega, 0]$ in M_ω is given by $\Psi^{-1}(0)/U(1)$, where $U(1)$ is the stabiliser of $[A_\omega, 0]$ in the gauge group, and

$$\Psi\colon \mathbb{C}^{k+r} \longrightarrow \mathbb{C}^r$$

is a $U(1)$–equivariant map with $\Psi(0) = 0$ which is transverse to $0 \in \mathbb{C}^r$ away from the origin in $\mathbb{C}^{k+r}$. The $U(1)$–action is the standard one on $\mathbb{C}^{k+r}$ and $\mathbb{C}^r$.

The moduli space M^*_ω of irreducible solutions to the monopole equations has one or several ends converging to the unique reducible solution because M_ω is compact. These ends are described by the restriction

$$\Psi\colon \mathbb{C}^{k+r} \setminus \{0\} \longrightarrow \mathbb{C}^r \ ,$$

which descends to a section of the bundle $H^{\oplus r}$ over the punctured cone $c(\mathbb{C}P^{k+r-1})$ associated with the $U(1)$–action. This shows that for a generic value of the cone parameter, the cross–section of the end is a smooth manifold N of dimension $2k-2$ whose fundamental class is the same as that of $\mathbb{C}P^{k-1} \subset \mathbb{C}P^{k+r-1}$. Truncating the cone at this generic value, we obtain a compact smooth manifold, also denoted M_ω, with boundary N.

The basepoint–fibration β over the moduli space restricts to N such that $\langle w_2^{k-1}(\beta), [N]\rangle = \langle w_2^{k-1}(\beta), [\mathbb{C}P^{k-1}]\rangle \neq 0$. This contradicts the fact that M_ω gives a zero–bordism of β, as a bundle over N, because zero–bordant bundles have vanishing Stiefel–Whitney numbers.

We have shown that Assumption 4.9 leads to a contradiction. Theorem 4.4 now follows from the next Proposition which asserts that Assumption 4.9 can be satisfied if and only if the intersection form of X is not diagonalisable over the integers.

PROPOSITION 4.10. (Elkies) *A negative definite unimodular symmetric bilinear form Q admits a characteristic vector v with $v^2 > -\operatorname{rk}(Q)$ if and only if Q is not diagonalisable over $\mathbb{Z}$.*

One direction is obvious: if Q is diagonalisable, then the sum of the basis vectors with respect to which Q is diagonal is a vector v with v^2 maximal among all characteristic vectors, and $v^2 = -\operatorname{rk}(Q)$. The converse was proved by Elkies in [**9**]. Note that for specific examples of non–diagonalisable definite forms, say even ones, one does not need to appeal to Elkies's proposition, as the required characteristic vector is easily found explicitely.

There are a number of possible variations of the argument in the proof of Theorem 4.4:

REMARK 4.11. Instead of arguing with the Kuranishi map $\Psi\colon \mathbb{C}^{k+r} \to \mathbb{C}^r$ for an unkown r, one could reduce to a situation where $r = 0$, so that $N = \mathbb{C}P^{k-1}$. This can be done by showing that for generic ω the Dirac operator $D^+_{A_\omega}$ is surjective.

REMARK 4.12. Using the orientability of the moduli space M^*_ω, the final contradiction is a consequence of Stokes's theorem applied to the top power of the Euler class of the basepoint–fibration. Thus, at the expense of invoking orientability, one does not need to use the vanishing of the Stiefel–Whitney numbers of zero–bordant bundles.

2.1. Comparison with Donaldson's proof. The above proof of Theorem 4.4 is very indirect, and the reader might be wondering how Donaldson ever thought of the statement. He found the statement, and the original proof, using Yang–Mills theory which gives an approach which, though more direct, is technically harder to carry out.

The model situation is that of the 4–sphere appearing as the boundary of the so–called one–instanton moduli space, which happens to be the 5–ball. Donaldson generalised this to prove that if X is a smooth closed oriented 4–manifold with negative definite intersection form Q_X, then the moduli space of ASD connections on the $SU(2)$–bundle with second Chern class 1 over X, though not compact, could be truncated to give a compact 5–dimensional space with X as its boundary. This space is a manifold away from some singularities arising from reducible connections, and thinking of it as a cobordism between X and the union of the links of the singularities gives the desired diagonalisation of Q_X.

This is a very natural, direct, argument, and fits in with the cobordism–invariance of the signature. It does not require any algebraic result like Proposition 4.10. However, the argument does require the cobordism to be oriented, whereas the indirect approach with the monopole moduli spaces does not need any orientations.

3. Proof of Theorem 4.5

Theorem 4.5 can be formulated in the following equivalent way:

THEOREM 4.13. *Let X be a smooth closed oriented spin 4–manifold. If $\min\{b_2^+(X), b_2^-(X)\} \leq 2$, then $\sigma(X) = 0$.*

It is clear that this statement would be false in the case of manifolds with $min\{b_2^+, b_2^-\} = 3$, because of the existence of the $K3$ surface.

In outline, the proof of Theorem 4.5 follows that of Theorem 4.4 given in the previous section. However, there is one new ingredient, which is the symmetry τ of the monopole equations for Spin- rather than Spin^c–structures. A variant of the proof will be used in the next Chapter to calculate the monopole invariants of the $K3$ surface.

Homotopy K3? Mention τ?

For the proof of the Theorem we may assume $b_1 = 0$, because of Lemma 4.7. Choosing the appropriate orientation and forming the connected sum with $S^2 \times S^2$ if necessary, we can assume $b_2^+ = 2$. If the signature is not zero, then by Rokhlin's theorem 1.15 it is $-16k$ for some $k > 0$. We will show that $k > 0$ leads to a contradiction.

We consider the monopole equations for a connection and a spinor for a Spin^c–structure coming from an honest spin structure on X. In this case $c_1(V_\pm) = 0$, and there is a reducible solution $(0, 0)$ to the unperturbed monopole equations (2.1) and (2.3), meaning that the connection $\hat{A}$ on $det_{\mathbb{C}} V_\pm$ is a product connection and the spinor ϕ vanishes identically. This solution is unique up to gauge equivalence. It gives a point in a moduli space M of dimension

$$\dim M = -\frac{1}{4}(2\chi(X) + \sigma(X)) = 4k - 3 \ .$$

The linearised equation at the reducible solution uncouples into the half de Rham complex (1.2) and the Dirac operator. The half de Rham complex does not contribute to $H^1_{(A_\omega,0)}$, and contributes a copy of $\mathbb{R}^2$ to $H^2_{(A_\omega,0)}$. The complex index of the Dirac operator D_A is $2k$, so if its cokernel is $\mathbb{C}^r$ for some $r \geq 0$, its kernel is $\mathbb{C}^{2k+r}$. Thus, the Kuranishi model of a neighbourhood of $[0,0]$ in M is given by $\Psi^{-1}(0)/U(1)$, where $U(1)$ is the stabiliser of $[0,0]$ in the gauge group, and

$$\Psi\colon \mathbb{C}^{2k+r} \longrightarrow \mathbb{C}^r \oplus \mathbb{R}^2$$

is a $U(1)$–equivariant map with $\Psi(0) = 0$ which is transverse to $0 \in \mathbb{C}^r \oplus \mathbb{R}^2$ away from the origin in $\mathbb{C}^{2k+r}$. The $U(1)$–action is the standard one on the spaces $\mathbb{C}^{2k+r}$ and $\mathbb{C}^r$ of harmonic spinors, and is trivial on the space $\mathbb{R}^2$ of self–dual harmonic two–forms.

The moduli space M^* of irreducible solutions to the monopole equations has some number of ends, all converging to the unique reducible solution because M is compact. These ends are described by the restriction

$$\Psi\colon \mathbb{C}^{2k+r} \setminus \{0\} \longrightarrow \mathbb{C}^r \oplus \mathbb{R}^2 \ ,$$

which descends to a section of the bundle $H^{\oplus r} \oplus \mathbb{R}^2$ over the punctured cone $c(\mathbb{C}P^{2k+r-1})$ associated with the $U(1)$–action.

Let $e \in H^2(\mathcal{B}^*, \mathbb{Z})$ be the Euler class of the basepoint–fibration β. We choose a smooth submanifold $\mathcal{V}$ of $\mathcal{B}^*$ of codimension $4k - 4$ representing e^{2k-2}, such that $\mathcal{V}$ intersects the link of $\mathcal{B} \setminus \mathcal{B}^*$ in a standard $\mathbb{C}P^{2k-2} \subset \mathbb{C}P^\infty$. If $\mathcal{V}$ is generic enough, its intersection with M^* is a smooth 1–dimensional manifold with some number of ends all aproaching the reducible solution. The number of ends is the same, modulo 2, as the top Stiefel–Whitney number of $H^{\oplus r} \oplus \mathbb{R}^2 \to \mathbb{C}P^{r+1}$, which is $= 0$ because of the trivial summand.

So far, this argument has not produced any contradictions. A contradiction could only have occurred if the Euler number had been odd. Then M^* would have been a 1–dimensional manifold with an odd number of ends. However, we have not really used the fact that X is spin, we could have made all these arguments with a Spin^c–structure on the connected sum of two copies of the Enriques surface, which clearly exists!

We use the assumption that we are dealing with a Spin–structure by using the symmetry τ introduced in Section 2 of Chapter 2. Recall that τ interchanges a Spin^c–structure with its conjugate, and therefore preserves the Spin^c–structures induced from honest Spin–structures. For such a Spin^c–structure τ acts as an involution on $\mathcal{B}$, preserving the moduli space M. This action fixes the gauge equivalence classes of reducible configurations and is free on $\mathcal{B}^*$.

Thus we can consider M^*/τ instead of M^*, and intersect this with $\mathcal{V}/\tau$, by arranging $\mathcal{V}$ to be τ–invariant. The number of ends in $(M^* \cap \mathcal{V})/\tau$ is the same, modulo 2, as the top Stiefel–Whitney number of $(H/\tau)^{\oplus r} \oplus \epsilon^{\oplus 2} \to \mathbb{C}P^{r+1}/\tau$, where ϵ is the non–trivial real line bundle associated with τ. After cutting down we are looking at $\epsilon^{\oplus 2} \to \mathbb{R}P^2$ for which $w_1^2 \neq 0$.

We have now reached the desired contradiction, because M^*/τ is a 1–dimensional manifold with an odd number of ends, which is impossible.

This completes the proof of Theorem 4.5.

3.1. Comparison with Donaldson's proof. Donaldson's [**6, 7**] approach to Theorem 4.5 was a natural generalisation of his proof of Theorem 4.4 – just as the proof we have given here is a natural extension of the proof of Theorem 4.4 using the monopole moduli spaces. However, the parallels go further: in both approaches, with instantons and with monopoles, Theorem 4.4 is proved using the rational cohomology of a moduli space truncated around the reducible solutions. Also in both approaches, Theorem 4.5 is proved relying on more subtle torsion information on the moduli spaces, which is available only when the base manifold is spin, viz. our use of τ and Donaldson's use of 2–torsion classes on moduli spaces.

4. Proof of Theorem 4.6

!!!TO BE WRITTEN!!!

Notes:

1. The $\frac{11}{8}$–Conjecture, for simply connected manifolds, made its appearance in print in Matsumoto [**24**], although other workers in the field also arrived at it around the same time.
2. Donaldson's proof of Theorem 4.4 in [**5, 7**] was the first application of gauge theory, in the guise of the Yang–Mills equations, to four–dimensional topology. In October/November 1994, many people noticed that the monopole equations give a simple proof for the case of even forms, and that this proof would give the full result if the algebraic Proposition 4.10 were true. In response to a question from Mrowka, Elkies [**9**] proved Proposition 4.10 in January/February 1995.
3. Theorem 4.5 was proved by Donaldson [**6, 7**], using Yang–Mills theory, with the additional assumption that $H_1(X,\mathbb{Z})$ be free of 2–torsion. Efforts to remove this assumption from his argument were unsuccessful, cf. [**27**]. The proof given here using the monopole moduli spaces was found in November 1994.
4. The ideas of subsection 1.1, especially those behind Example 4.8, are pursued systematically in [**14**].