CHAPTER 3

Bounds on solutions and their applications

In this Chapter we shall see that the Weitzenböck decomposition of the Dirac Laplacian

$$D_A^- D_A^+ = \nabla_A^* \nabla_A + \frac{1}{4} s_g + \frac{1}{2} \gamma(F_{\hat{A}}^+)$$

implies strong bounds on solutions to the monopole equations. These bounds severely constrain the possible solutions, and imply in particular that the moduli spaces are always compact.

These results have no analogues for most other equations of this type, be they the anti–self–dual Yang–Mills equations, or variants of the monopole equations involving higher rank connections or several spinors, or the equations for pseudo–holomorphic curves.

1. Estimates from the Weitzenböck formula

Let X be a closed oriented 4–manifold with a fixed Riemannian metric g and a fixed Spin^c–structure $\tilde{P}$.

The most fundamental bound on solutions (A, ϕ) of the monopole equations (2.1) and (2.3) is a C^0 bound on ϕ obtained from the maximum principle. Recall that all our configurations are of Sobolev class L_4^2, and so are continuous. At a point where $|\phi|$ achieves its maximum, we calculate using the Weitzenböck formula, the equations and Lemma 2.3:

$$\begin{aligned} 0 &\le \frac{1}{2}\Delta|\phi|^2 = \langle \nabla_A^* \nabla_A \phi, \phi\rangle - |\nabla_A \phi|^2 \le \langle \nabla_A^* \nabla_A \phi, \phi\rangle \\ &= \langle D_A^- D_A^+ \phi, \phi\rangle - \frac{1}{4} s_g |\phi|^2 - \frac{1}{2}\langle \gamma(F_{\hat{A}}^+)\phi, \phi\rangle \\ &= -\frac{1}{4} s_g |\phi|^2 - \frac{1}{2}\langle \gamma(\sigma(\phi,\phi))\phi, \phi\rangle = -\frac{1}{4} s_g |\phi|^2 - 2|\sigma(\phi,\phi)|^2 \\ &= -\frac{1}{4} s_g |\phi|^2 - \frac{1}{4}|\phi|^4 \ . \end{aligned}$$

Thus, either ϕ vanishes identically and all solutions are reducible, or the scalar curvature function s_g is negative somewhere and $|\phi|^2$ is bounded above, pointwise, by the maximum of $-s_g$ over X.

Notice that, using Lemma 2.3 again, the curvature equation (2.3) implies

$$|F_{\hat{A}}^+| = |\sigma(\phi,\phi)| = \frac{1}{2\sqrt{2}}|\phi|^2 \ ,$$

so that the C^0 bound on ϕ also gives a C^0 bound on $F_{\hat{A}}^+$.

It is an easy exercise to carry out the above calculation with the perturbed curvature equation (2.5) instead of the unperturbed one (2.3). One finds that for a solution (A,ϕ) of equations (2.4) and (2.5), the following estimate holds at a point where $|\phi|$ is maximal:

$$\begin{aligned} 0 \leq \frac{1}{2}\Delta|\phi|^2 &\leq \langle D_A^- D_A^+\phi,\phi\rangle - \frac{1}{4}s_g|\phi|^2 - \frac{1}{2}\langle\gamma(F_{\hat{A}}^+)\phi,\phi\rangle \\ &= -\frac{1}{4}s_g|\phi|^2 - \frac{1}{2}\langle\gamma(\sigma(\phi,\phi)+\omega)\phi,\phi\rangle \\ &= -\frac{1}{4}s_g|\phi|^2 - \frac{1}{4}|\phi|^4 - 2\langle\omega,\sigma(\phi,\phi)\rangle \\ &\leq -\frac{1}{4}s_g|\phi|^2 - \frac{1}{4}|\phi|^4 + 2|\omega|\cdot|\sigma(\phi,\phi)| \\ &= -\frac{1}{4}s_g|\phi|^2 - \frac{1}{4}|\phi|^4 + \frac{1}{\sqrt{2}}|\omega|\cdot|\phi|^2 \ . \end{aligned}$$

As above, this also gives a bound on $F_{\hat{A}}^+$. We have proved:

PROPOSITION 3.1. *Let $s_{g,\omega} = min\{0,\ s_g - 2\sqrt{2}|\omega|\}$ with the minimum taken over all of X, and let (A,ϕ) be a solution to the monopole equations* (2.4) *and* (2.5). *Then ϕ and $F_{\hat{A}}^+$ are pointwise bounded as follows:*

$$|\phi|^2 \leq -s_{g,\omega} \ , \tag{3.1}$$

$$|F_{\hat{A}}^+| \leq -\frac{1}{2\sqrt{2}}s_{g,\omega} + max|\omega| \ . \tag{3.2}$$

These bounds are good as ω is in L_3^2 and therefore continuous.

In addition to these pointwise bounds, we also have the following integral bounds on all solutions:

PROPOSITION 3.2. *For every solution (A,ϕ) to the monopole equations* (2.4) *and* (2.5) *we have:*

$$||\phi||_{L^4}^2 \leq ||(-s_g + 2\sqrt{2}|\omega|)||_{L^2} \ , \tag{3.3}$$

$$||\nabla_A\phi||_{L^2} \leq \frac{1}{2}||(-s_g + 2\sqrt{2}|\omega|)||_{L^2} \ , \tag{3.4}$$

$$||F_{\hat{A}}^+||_{L^2} \leq \frac{1}{2\sqrt{2}}||(-s_g + 2\sqrt{2}|\omega|)||_{L^2} + ||\omega||_{L^2} \ . \tag{3.5}$$

PROOF. We apply both sides of the Weitzenböck formula to ϕ and take the L^2 inner product with ϕ. After substituting from the equations, we obtain:

$$0 = \int_X (|\nabla_A \phi|^2 + \frac{1}{4} s_g |\phi|^2 + \frac{1}{4}|\phi|^4 + 2\langle \omega, \sigma(\phi,\phi)\rangle) dvol_g \ . \tag{3.6}$$

This shows

$$\begin{aligned} \int_X |\phi|^4 dvol_g &\leq \int_X (-s_g|\phi|^2 - 8\langle \omega, \sigma(\phi,\phi)\rangle) dvol_g \\ &\leq \int_X (-s_g|\phi|^2 + 8|\omega| \cdot |\sigma(\phi,\phi)|) dvol_g \\ &= \int_X |\phi|^2(-s_g + 2\sqrt{2}|\omega|) dvol_g \\ &\leq (\int_X |\phi|^4 dvol_g)^{\frac{1}{2}} \cdot (\int_X (-s_g + 2\sqrt{2}|\omega|)^2 dvol_g)^{\frac{1}{2}} \ , \end{aligned}$$

which implies (3.3).

Starting again with (3.6), proceeding as above and using the inequality we just proved, (3.4) follows.

To prove (3.5) we calculate

$$\begin{aligned} \int_X |F_{\hat{A}}^+|^2 dvol_g &= \int_X |\sigma(\phi,\phi) + \omega|^2 dvol_g \\ &\leq \int_X (\frac{1}{8}|\phi|^4 + \frac{1}{\sqrt{2}}|\phi|^2|\omega| + |\omega|^2) dvol_g \\ &= \int_X (\frac{1}{2\sqrt{2}}|\phi|^2 + |\omega|)^2 dvol_g \ . \end{aligned}$$

Now the claim follows from (3.3). □

REMARK 3.3. Specialising to the case $\omega = 0$, i.e. the unperturbed curvature equation, we find another proof of Corollary 2.4. This should be no surprise as these manipulations with the Weitzenböck formula are the same as the ones we carried out in the proof of Proposition 2.2.

These bounds imply that on a given Riemannian 4–manifold there tend to be at most finitely many Spin^c–structures admitting solutions to the monopole equations:

COROLLARY 3.4. *Let X be a closed oriented 4–manifold. For fixed parameters $(g,\omega) \in \mathcal{P}$ there are at most finitely many* Spin^c*–structures on X for which M_ω is not empty, and of non–negative expected dimension.*

PROOF. The assumption of non–negative expected dimension means that

$$c_1^2(V_\pm) \geq 2\chi(X) + 3\sigma(X) .$$

Thus $c_1^2(V_\pm)$ is bounded below, and the Chern–Weil formula $||F_{\hat{A}}^-||_{L^2}^2 = ||F_{\hat{A}}^+||_{L^2}^2 - 4\pi^2 c_1^2(V_\pm)$ shows that the L^2 bound on $F_{\hat{A}}^+$ implies an L^2 bound for $F_{\hat{A}}^-$.

Thus the projections of $F_{\hat{A}}$ to the harmonic subspaces $\mathcal{H}_+^2$ and $\mathcal{H}_-^2$ are both bounded, so that the image of $c_1(V_\pm)$ in $H^2(X,\mathbb{R}) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$ is contained in a bounded set. Thus there are only finitely many possibilities for $c_1(V_\pm)$, and therefore only finitely many Spin^c–structures for which there are solutions. □

There is a useful variant of this result:

COROLLARY 3.5. *Assume that $b_2^+(X) > 0$, and that $(g,\omega) \in \mathcal{P}$ is generic. Then there are at most finitely many* Spin^c*–structures for which M_ω is nonempty.*

PROOF. By the transversality result of Section 6 in Chapter 2, the assumption $b_2^+ > 0$ implies that for generic parameters there are only irreducible solutions, and these solutions are transverse zeroes of f_ω, so that the only non–empty moduli spaces have non–negative expected dimension. □

2. Compactness of the moduli space

We shall now show that the moduli space M of solutions to the monopole equations modulo gauge equivalence is compact. The idea of the proof is rather simple, and is a routine application of the inequalities proved in the previous Section.

We fix a closed oriented Riemannian four–manifold (X,g) and a Spin^c structure on it. For simplicity we shall first discuss the unperturbed equations (2.1) and (2.3).

Let (A,ϕ) be a solution of equations (2.1) and (2.3), and let A_0 be a fixed reference connection. Recall that (3.5) gives an L^2 bound on $F_{\hat{A}}^+$, and that this together with the Chern–Weil formula also gives an L^2 bound on $F_{\hat{A}}^-$, so that the total curvature of $\hat{A}$ is bounded in L^2. The map from $\mathcal{A}/\mathcal{G}$ to curvature two–forms of the induced connections on $det_{\mathbb{C}}(V_\pm)$ has compact fibers isomorphic to $J(X)$. Thus the L^2 bound on the curvature gives an L^2 bound on $\hat{A}$, and the projection

$$\pi\colon M \longrightarrow \mathcal{A}/\mathcal{G}$$
$$[A,\phi] \longmapsto [A]$$

is bounded in L^2.

By applying gauge transformations in the component of the identity, we can achieve

$$d^*(A - A_0) = 0 \ .$$

Thus $A - A_0 = h + \alpha$, with h harmonic and $\alpha = d^*\beta$. Using all components of the gauge group, we may assume that the harmonic part h is bounded (in any norm), because $J(X)$ is compact.

As α is orthogonal to the kernel of the elliptic operator $d^* + d^+$ on 1–forms, there exists a constant c such that

(3.7)
$$||\alpha||_{L^2_1} \leq c||(d^* + d^+)\alpha||_{L^2} = c||d^+\alpha||_{L^2} = c||F^+_{\hat{A}} - F^+_{\hat{A}_0}||_{L^2} \leq c||F^+_{\hat{A}}||_{L^2} + c' \ .$$

Thus $A - A_0 = h + \alpha$ is bounded in L^2_1. By the Rellich lemma the inclusion $L^2_1 \to L^2$ is compact. Thus π is compact in the L^2 topology.

The fiber of π over a point $[A]$ is contained in the quotient by $U(1)$ of a subset in $ker(D^+_A)$. This vector space is finite–dimensional, and the relevant subset is bounded because of the C^0 estimate (3.1) on ϕ. Thus π has compact fibers. We conclude that M is compact in the L^2 topology.

We shall now prove a more precise result, in which all the derivatives of solutions are controlled as well. For this we need to make use again of the Sobolev multiplication theorem [**26**], which we already recalled and used in Section 2 of Chapter 2.

THEOREM 3.6. *Let (A_i, ϕ_i) be a sequence of L^2_4 solutions to the monopole equations* (2.1) *and* (2.3). *There is a sequence of L^2_5 gauge transformations u_i such that the $u_i(A_i, \phi_i)$ are smooth and have a subsequence converging in the C^∞ topology to a C^∞ solution (A, ϕ).*

In particular, M is compact in the C^∞ topology.

PROOF. As above, after applying gauge transformations we can write $A = A_0 + h + \alpha$, with A_0 fixed, h harmonic, and α in the image of d^*; and similarly for $A_i = A_0 + h_i + \alpha_i$. The h_i are solutions of a linear elliptic equation, and we may assume that they are bounded in C^∞. Thus they have a convergent subsequence.

Recall that for every solution (A, ϕ), we have a priori bounds on $||\phi||_{L^4}$, on $||\nabla_A\phi||_{L^2}$, on $||\phi||_{C^0}$, and on $||F^\pm_{\hat{A}}||_{L^2}$. When bounding the derivatives, it will be important to keep in mind that the norms are defined with respect to a fixed background connection, say A_0, and not with respect to the variable A.

As in (3.7), we have

(3.8)
$$||\alpha||_{L^p_{k+1}} \leq c||(d^* + d^+)\alpha||_{L^p_k} = c||d^+\alpha||_{L^p_k} = c||\sigma(\phi,\phi) - F^+_{\hat{A}_0}||_{L^p_k} \leq c||\sigma(\phi,\phi)||_{L^p_k} + c\ .$$

Here and in the rest of this proof, all constants are denoted by c. For $k = 0$, the last term is bounded because of the C^0 bound on ϕ.

The Dirac equation $D^+_A\phi = 0$ is equivalent to $D^+_{A_0}\phi = -\gamma(a)\phi$, where $a = A - A_0 = h + \alpha$. As h is bounded in C^∞, bounding the derivatives of a is the same as bounding those of α.

The basic elliptic estimate for the Dirac operator $D^+_{A_0}$ gives:

$$\begin{aligned} ||\phi||_{L^2_2} &\leq c(||D^+_{A_0}\phi||_{L^2_1} + ||\phi||_{L^2}) \leq c||\gamma(a)\phi||_{L^2_1} + c \\ &\leq c||\gamma(a)\phi||_{L^2} + c||\gamma(da)\phi||_{L^2} + c||\gamma(a)\nabla_{A_0}\phi||_{L^2} + c\ , \end{aligned}$$

where again c stands for all constants. The first and second terms are bounded because of the bounded Sobolev multiplication $L^4 \times L^4 \to L^2$, using (3.8) to bound a in L^4_1. The last term can be estimated as

$$||\gamma(a)\nabla_{A_0}\phi||_{L^2} \leq c||a||_{L^p_1}||\nabla_A\phi - a\phi||_{L^2}\ ,$$

for some $p > 4$, where the first term on the right–hand–side is bounded because of (3.8), and the second is bounded because $\nabla_A\phi$ and $\gamma(a)\phi$ have already been bounded in L^2.

By the bounded Sobolev multiplication $L^2_2 \times L^2_2 \to L^3_1$, the L^2_2 bound on ϕ gives an L^3_1 bound on $\sigma(\phi,\phi)$. Using this in (3.8), we have an L^3_2 bound on α. Using this bound, we also have an L^3_1 bound on $\gamma(a)\phi$, and therefore an L^3_2 bound on ϕ, as in the argument above.

Now, $L^3_2 \times L^3_2 \to L^2_2$ is easily bounded, so that the same arguments as above give L^2_3 bounds for both α and ϕ.

We now have L^2_3 bounds on a and on ϕ, which we can improve to L^2_k bounds for all k by induction. Suppose we already have L^2_k bounds for some $k \geq 3$. Then the bounded Sobolev multiplication $L^2_k \times L^2_k \to L^2_k$ gives an L^2_k bound on $D_{A_0}\phi$, and thus an L^2_{k+1} bound on ϕ. In the same way, we have an L^2_k bound on $F^+_{\hat{A}} = \sigma(\phi,\phi)$, giving an L^2_{k+1} bound on a via (3.8).

Thus, the sequence $u_i(A_i,\phi_i)$ must have a subsequence converging with all its derivatives to a smooth limit, which is also a solution to the monopole equations. ☐

As a corollary of this proof we have:

COROLLARY 3.7. *The moduli spaces of L^2_k solutions to the monopole equations* (2.1) *and* (2.3) *modulo L^2_{k+1} gauge equivalence for the different $k \geq 4$ are all diffeomorphic to each other, and to the moduli space of smooth solutions.*

2.1. Two generalisations. An elaboration of the proof of Theorem 3.6 yields the following stronger result:

THEOREM 3.8. *The functional $E\colon \mathcal{B} \to \mathbb{R}$ defined in Definition 2.1 satisfies the Palais–Smale condition.*

Given the bounds on solutions of the perturbed monopole equations (2.4) and (2.5) proved in Propositions 3.1 and 3.2, the proof of the compactness theorem 3.6 goes through for the moduli space M_ω for any ω. However, even after gauge transformations, the solutions representing points in M_ω are only as regular as ω.

Notes:

1. Witten [**37**] was the first to exploit the interplay between the monopole equations and the Weitzenböck formula, and Kronheimer and Mrowka [**18**] first indicated how to derive the compactness of the moduli space from this.
2. A proof of Theorem 3.8 was published in [**13**].