CHAPTER 1

Four-dimensional linear algebra and bundle theory

This Chapter contains a review of standard material needed in later chapters for the study of the monopole equations. We first discuss the notion of self-duality of 2-forms, which is a key feature of four-dimensional geometry. We then review Clifford algebra, Spin^c-structures and Dirac operators. Unlike the self-duality of 2-forms, these concepts are not special to dimension 4, although we restrict ourselves to this case.

1. Self-duality

Let V be an oriented n-dimensional vector space with an inner product g, or $\langle\ ,\ \rangle$. The *Hodge star operator*

$$*\colon \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)$$

is characterised by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dvol_g .$$

The Hodge star preserves inner products, and satisfies $** = (-1)^{k(n-k)}$. In the case $n = 4$ and $k = 2$ therefore, we have $** = +1$, and $\Lambda^2(V^*)$ splits as the orthogonal direct sum of the ± 1 eigenspaces of $*$:

$$\Lambda^2(V^*) = \Lambda^2_+(V^*) \oplus \Lambda^2_-(V^*)\ . \tag{1.1}$$

If an inner product $\tilde{g}$ differs from g only by a scale factor, then it determines the same splitting. Changing the orientation of V interchanges $\Lambda^2_+(V^*)$ and $\Lambda^2_-(V^*)$. In terms of an oriented orthonormal basis $e_0, \dots, e_3$ for V^*, the summand $\Lambda^2_+(V^*)$ is the 3-dimensional subspace of $\Lambda^2(V^*)$ spanned by the vectors

$$\begin{aligned} &e_0 \wedge e_1 + e_2 \wedge e_3\ ,\\ &e_0 \wedge e_2 - e_1 \wedge e_3\ ,\\ &e_0 \wedge e_3 + e_1 \wedge e_2\ . \end{aligned}$$

Now let (X, g) be an oriented Riemannian four-manifold. Applying the above construction to the tangent space at each point, we obtain a decomposition of the space of 2-forms on X into the *self-dual* and *anti-self-dual* forms:

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X).$$

Because the formal adjoint d^* of the exterior differential d is $\pm * d*$, the space of closed and co-closed 2-forms

$$\mathcal{H}^2 = \{\, \alpha \mid d\alpha = 0,\ d^*\alpha = 0 \,\}$$

is mapped to itself by the star operator, and we have an induced splitting

$$\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-.$$

On a compact manifold, $\mathcal{H}^2$ is the space of harmonic 2-forms and is identified with $H^2(X,\mathbb{R})$ by Hodge theory, so we can regard this splitting as a metric-dependent decomposition of the second cohomology. Seen this way, $\mathcal{H}^+$ and $\mathcal{H}^-$ are maximal positive and negative subspaces for the quadratic form on $H^2(X,\mathbb{R})$ defined by the cup product:

$$\begin{aligned} H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (a,b) \qquad\qquad &\longmapsto (a \smile b)[X]\ . \end{aligned}$$

It follows in particular that the dimensions of $\mathcal{H}^+$ and $\mathcal{H}^-$ are independent of the metric. These we denote as usual by $b_2^+(X)$ and $b_2^-(X)$; their sum is the second Betti number, and their difference is the *signature* of X:

$$\begin{aligned} b_2(X) &= b_2^+(X) + b_2^-(X), \\ \sigma(X) &= b_2^+(X) - b_2^-(X). \end{aligned}$$

We will denote the self-dual and anti-self-dual parts of a 2-form using $\pm$ subscripts. If $\alpha \in \Omega^1$, we write $d^\pm\alpha$ for $(d\alpha)_\pm$.

PROPOSITION 1.1. *If X is closed, then*

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^2_+ \tag{1.2}$$

is a complex whose three cohomology groups are isomorphic to $H^0(X,\mathbb{R})$, $H^1(X,\mathbb{R})$ and $\mathcal{H}^2_+$. In particular, the index of (1.2) *is*

$$b_0(X) - b_1(X) + b_2^+(X) = \frac{1}{2}(\chi(X) + \sigma(X))\ ,$$

where χ is the Euler characteristic and σ is the signature.

This, together with the Dirac equation, is the model for the elliptic complex we will encounter when linearizing the nonlinear monopole equations.

PROOF. For $H^0(X,\mathbb{R})$ the claim is clear. For the cohomology at the middle term, observe that for all $\alpha \in \Omega^1$ we have

$$\int_X (|d^+\alpha|^2 - |d^-\alpha|^2) dvol_g = \int_X d\alpha \wedge d\alpha = \int_X d(\alpha \wedge d\alpha) = 0$$

by Stokes's theorem. Thus $d^+\alpha = 0$ if and only if $d^-\alpha = 0$. This proves that the first cohomology of the complex is $H^1(X,\mathbb{R})$.

For the last term, note that for $h \in \mathcal{H}^2_+$ we have

$$\int_X \langle h, d^+\alpha\rangle dvol_g = \int_X h \wedge d^+\alpha = \int_X h \wedge d\alpha = \int_X d(h \wedge \alpha) = 0,$$

again by Stokes's theorem. Thus $\mathcal{H}^2_+$ is L^2-orthogonal to $Im(d^+)$ and injects into $\Omega^2_+/Im(d^+)$. We want to show that this injection is surjective as well. Let $\omega \in \Omega^2_+$ have Hodge decomposition

$$\omega = h + d\alpha + *d\beta \ ,$$

with $h \in \mathcal{H}^2$. Applying the star operator, we obtain

$$\omega = *h + d\beta + *d\alpha \ .$$

The uniqueness of the Hodge decomposition shows that h is self-dual and $d\alpha = d\beta$. Thus $\omega = h + d^+(2\alpha)$, and $h \in \mathcal{H}^2_+$ represents ω in the quotient $\Omega^2_+/Im(d^+)$.

The index of the complex (the alternating sum of the dimensions of the cohomology groups) can be expressed in terms of χ and σ as stated in the proposition because $\sigma = b_2^+(X) - b_2^-(X)$ and

$$\begin{aligned}\chi &= b_0 - b_1 + b_2 - b_3 + b_4 \\ &= 2b_0 - 2b_1 + b_2,\end{aligned}$$

by Poincaré duality. □

2. Spin^c-structures

2.1. Clifford modules. The standard Dirac operator on $\mathbb{R}^n$ is a particular self-adjoint operator whose square is the Laplace operator. Consider a first-order differential operator D of the form

$$D = \sum_{i=1}^{n} A_i \frac{\partial}{\partial x^i} \tag{1.3}$$

acting on $\mathbb{C}^N$-valued functions on $\mathbb{R}^n$, where the A_i are constant N-by-N matrices. This operator is formally self-adjoint if the A_i are skew-adjoint. The Laplacian acting on $\mathbb{C}^N$-valued functions is the second-order operator

$$\Delta = -\sum_{i=1}^{n} I_N(\frac{\partial}{\partial x^i})^2,$$

where I_N is the identity matrix, so the condition $D^2 = \Delta$ is

$$A_i^2 = -I_N \quad \text{and} \quad A_iA_j + A_jA_i = 0. \tag{1.4}$$

In the presence of the first of these two relations, the condition that A_i is skew-adjoint is equivalent to it being unitary.

This situation can be abstracted as follows. Let H be a finite-dimensional real vector space with inner product g. A *Clifford module* for H is a complex vector space V with Hermitian inner product and a linear map

$$\gamma\colon H \to \mathrm{End}(V),$$

satisfying the relations

$$\begin{gathered} \gamma(v)^* = -\gamma(v) \\ \gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2g(v,w)\mathbf{1}_V \ . \end{gathered} \tag{1.5}$$

The elements of V are called *spinors*.

The above relations ensure that the linear map γ extends to an algebra homomorphism of the Clifford algebra of (H, g), denoted $\mathrm{Cl}(H)$, which is defined to be the quotient of the tensor algebra of H by the relation

$$v \otimes w + w \otimes v + 2g(v,w)\mathbf{1}_V = 0 \ .$$

As a vector space the Clifford algebra $\mathrm{Cl}(H)$ is isomorphic to the exterior algebra $\Lambda^*(H)$. The action of H on V via γ is called *Clifford multiplication*, as is the multiplication in the Clifford algebra itself.

Although we will not need to know the structure of Clifford modules in general for this book, we state here, without proof, this basic result.

PROPOSITION 1.2. *If $n = 2m$ is even then up to isomorphism there is a unique irreducible Clifford module (V, γ) for $\mathbb{R}^n$. Its dimension is 2^m. If $n = 2m+1$ is odd then there are two irreducible Clifford modules for $\mathbb{R}^n$. Both have dimension 2^m, and one is obtained from the other by replacing the Clifford multiplication γ with $-\gamma$.* □

Any finite-dimensional Clifford module is a sum of irreducible modules: since each $\gamma(v)$ is skew-adjoint, the orthogonal complement of a submodule is also a submodule.

Thus in the even dimensional case we will call any irreducible Clifford module a *standard Clifford module.*

In the odd dimensional case we can use an orientation of H distinguish its two irreducible Clifford modules. The inner product and an orientation define an oriented volume form $*1 = dvol$ which acts by Clifford multiplication as ± 1. Writing $n = 2m+1$, the Clifford modules can be written (V, γ) and $(V, -\gamma)$, with $V = \mathbb{C}^{2^m}$. Following the usual convention we will call an irreducible Clifford module a *standard Clifford module* if the volume form acts as $(-1)^{m+1}$.

2.1.1. *A standard Clifford module for* $\mathbb{R}^4$. We define an action of $\mathbb{R}^4$ on $\mathbb{C}^4$ as follows. Choose an oriented orthonormal basis, $e_0, \ldots, e_3$, for $\mathbb{R}^4$ and define

$$\gamma(e_j) = A_j = \begin{pmatrix} 0 & -B_j^* \\ B_j & 0 \end{pmatrix}$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad , B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One can check that γ extends to an isomorphism $\gamma : \mathrm{Cl}(\mathbb{R}^4) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$. Therefore the Clifford module we have defined is irreducible. From the basic theory of matrix algebras it follows that it is unique up to isomorphism.

As mentioned above, the Clifford algebra $\mathrm{Cl}(\mathbb{R}^4)$ is linearly isomorphic to $\Lambda^*(\mathbb{R}^4)$. The standard way to choose this isomorphism is so that

$$\gamma(e_{i_1} \wedge \ldots \wedge e_{i_l}) = \gamma(e_{i_1}) \ldots \gamma(e_{i_l})$$

whenever $i_1 < \ldots < i_l$. However, this does *not* mean that γ is a ring homomorphism, because if we multiply forms which are not orthogonal, we obtain different answers in the exterior and Clifford algebras.

From the definition of the A_j it is natural to decompose the space of spinors $\mathbb{C}^4 = \mathbb{C}^2_+ \oplus \mathbb{C}^2_-$ so that the volume form acts as -1 on $\mathbb{C}^2_+$ and 1 on $\mathbb{C}^2_-$. The elements of $\mathbb{C}^2_+$ (respectively $\mathbb{C}^2_-$) are called positive (respectively negative) spinors.

Under γ the self-dual two-forms

$$e_0 \wedge e_1 + e_2 \wedge e_3 \ ,$$
$$e_0 \wedge e_2 - e_1 \wedge e_3 \ ,$$
$$e_0 \wedge e_3 + e_1 \wedge e_2 \ ,$$

act non-trivially only on $\mathbb{C}^2_+$ as $2B_1, 2B_2$ and $2B_3$ respectively. In fact γ induces

$$\big(\Lambda^1(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)\big) \otimes \mathbb{C} \cong \mathrm{Hom}(\mathbb{C}^2_+, \mathbb{C}^2_-) \oplus \mathrm{Hom}(\mathbb{C}^2_-, \mathbb{C}^2_+)$$
$$\Lambda^2_\pm(\mathbb{R}^4) \otimes \mathbb{C} \cong \mathrm{End}_0(\mathbb{C}^2_\pm).$$

Here End_0 denotes the trace-free endomorphisms.

The fact that the space of complexified self-dual two-forms is isomorphic to the traceless endomorphisms of the positive spinors will be crucial in the formulation of the monopole equations. See the beginning of Chapter 2.

2.1.2. *A standard Clifford module for* $\mathbb{R}^3$. We define an action of $\mathbb{R}^3$ on $\mathbb{C}^2$ as follows. Choose an oriented orthonormal basis e_1, e_2, e_3 for $\mathbb{R}^3$

$$\gamma(e_j) = -B_j$$

with B_j as above. One can check that γ extends to an isomorphism $\gamma\colon \mathrm{Cl}(\mathbb{R}^3) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$. Note that

$$\gamma(e_1 \wedge e_2 \wedge e_3) = -B_1B_2B_3 = \mathbf{1}_{\mathbb{C}^2}\ ,$$

so this is a standard Clifford module.

2.1.3. *Conjugation of Clifford modules.* The sets of complex linear endomorphisms of a complex vector space, V, and its conjugate, $\bar{V}$, are the same. Therefore a Clifford module $\gamma\colon H \to \mathrm{End}(V)$ gives us another Clifford module $\gamma\colon H \to \mathrm{End}(\bar{V})$. In particular if $\gamma\colon H \to \mathrm{End}(V)$ is standard then so is $\gamma\colon H \to \mathrm{End}(\bar{V})$, and hence they are isomorphic.

2.2. Spin^c-structures on manifolds. In constructing a Dirac operator on a general Riemannian manifold, the first problem one encounters is topological. To begin with, let X be a paracompact Hausdorff space and $H \to X$ an oriented real vector bundle equipped with an inner product. (Later X will be an oriented Riemannian manifold and H will be its cotangent bundle.)

DEFINITION 1.3. A Spin^c-*structure*, $\mathfrak{s}$, on $H \to X$ is a pair (V, γ) where $V \to X$ is a complex vector bundle with a Hermitian metric and γ is a bundle map $\gamma : H \to \mathrm{End}(V)$ which is fiberwise a standard Clifford module.

It is easy to see, that existence or non-existence of a Spin^c-structure on a given $H \to X$ is indeed a purely topological problem, which we shall now discuss in the framework of Čech cohomology.

For a Lie group G we shall denote by $\mathcal{S}_G$ the sheaf assigning to an open subset of X the continuous functions on that set with values in G.

Let $\{U_a\}$ be a covering of X by open sets for which $H|_{U_a}$ is trivial for all a. We can then find Spin^c-structures $\mathfrak{s}_a$ for the restrictions to these open sets. Moreover, these Spin^c-structures are unique up to isomorphism. For each ordered pair (a, b), choose an isomorphism

$$\phi_{ab} \colon \mathfrak{s}_a|_{U_a \cap U_b} \to \mathfrak{s}_b|_{U_a \cap U_b} \ , \tag{1.6}$$

arranging that $\phi_{ba} = \phi_{ab}^{-1}$. If each triple-overlap function

$$\psi_{abc} = \phi_{ab}\phi_{bc}\phi_{ca}$$

is the identity on its domain of definition, then one may form a Spin^c-structure $\mathfrak{s}$ on $H \to X$ from the given collection. In general however, ψ_{abc} is an automorphism of $\mathfrak{s}_a|_{U_a \cap U_b \cap U_c}$. Since a standard Clifford module has automorphism group $U(1) = S^1$, ψ_{abc} must take values in S^1. We therefore have functions

$$\psi_{abc} \colon U_a \cap U_b \cap U_c \to S^1 \ . \tag{1.7}$$

These satisfy the cocycle condition on quadruple overlaps, and therefore define a 2-cocycle for the sheaf $\mathcal{S}_{S^1}$. Any other choice ϕ'_{ab} of isomorphisms is of the form $\phi'_{ab} = \pi_{ab}\phi_{ab}$, where π_{ab} is S^1-valued, with S^1 again acting pointwise as the automorphisms of $\mathfrak{s}_a$. The effect of choosing ϕ'_{ab} rather than ϕ_{ab} is to alter $\{\psi_{***}\}$ by the coboundary of the 1-cochain $\{\pi_{**}\}$. Thus the Čech cohomology class of the 2-cocycle depends only on $H \to X$ and not on the choices made; we write

$$\mathfrak{o}(H) = [\psi_{***}] \in H^2(X; \mathcal{S}_{S^1}) \tag{1.8}$$

for this class. It is the obstruction to the existence of a Spin^c-structure for $H \to X$. Its construction shows that it behaves naturally under pull-back, and is thus a characteristic class of SO(n)-bundles.

If a Spin^c-structure on $H \to X$ exists, one can similarly discuss the question of uniqueness. If $\mathfrak{s}$ and $\mathfrak{s}'$ are two Spin^c-structures, then we can find an open cover $\{U_a\}$ which trivializes both, and we can find isomorphisms $\tau_a : \mathfrak{s}|_{U_a} \to \mathfrak{s}'|_{U_a}$. The collection of functions $\sigma_{ab} = \tau_a^{-1}\tau_b$ is a 1-cocycle with values in $\mathcal{S}_{S^1}$. Any change in the τ_* changes this cocycle by a coboundary. So the cohomology class

$$\delta(\mathfrak{s}, \mathfrak{s}') = [\sigma_{**}] \in H^1(X; \mathcal{S}_{S^1}) \tag{1.9}$$

is independent of the choices made. It is the obstruction to finding an isomorphism between the Spin^c-structures.

Conversely, given an element $\delta \in H^1(X; \mathcal{S}_{S^1})$, and a Spin^c-structure $\mathfrak{s}$, one can construct $\mathfrak{s}'$ with $\delta(\mathfrak{s}, \mathfrak{s}') = \delta$. This may be done in the style of the above discussion, using transition functions.

Alternativley, note that the group $H^1(X; \mathcal{S}_{S^1})$ classifies principal bundles on X with structure group $\mathrm{U}(1)$ up to isomorphism, or equivalently complex line-bundles with Hermitian metric. To construct $\mathfrak{s}'$, define $V_{\mathfrak{s}'}$ as

$$V_{\mathfrak{s}'} = V_{\mathfrak{s}} \otimes L_\delta \ , \tag{1.10}$$

where L_δ is a Hermitian line-bundle corresponding to δ. There is then a canonical isomorphism $\iota\colon \mathrm{End}(V_{\mathfrak{s}}) \to \mathrm{End}(V_{\mathfrak{s}'})$, and one defines $\gamma_{\mathfrak{s}'}$ as the composite representation $\iota \circ \gamma_{\mathfrak{s}}$.

The classes $\mathfrak{o}(H)$ and $\delta(\mathfrak{s}, \mathfrak{s}')$ become computable if we relate the above discussion to integral cohomology. Consider the exact sequence of sheaves

$$\mathcal{S}_{\mathbb{Z}} \longrightarrow \mathcal{S}_{\mathbb{R}} \xrightarrow{\exp} \mathcal{S}_{S^1} \ .$$

It gives rise to a long exact sequence in cohomology, in which all the groups $H^i(X; \mathcal{S}_{\mathbb{R}})$ with $i > 0$ vanish because $\mathcal{S}_{\mathbb{R}}$ is a fine sheaf. So $H^1(X; \mathcal{S}_{S^1})$ is isomorphic to $H^2(X; \mathbb{Z})$: this is the statement that Hermitian line-bundles are classified up to isomorphism by their first Chern classes. *Mutatis mutandis*, the group $H^2(X; \mathcal{S}_{S^1})$ is isomorphic to $H^3(X; \mathbb{Z})$.

Our discussion of Spin^c-structures can be summarized in the following proposition.

PROPOSITION 1.4. *An* $\mathrm{SO}(n)$*-bundle* $H \to X$ *admits a* Spin^c*-structure if and only if the characteristic class* $\mathfrak{o}(H) \in H^3(X; \mathbb{Z})$ *vanishes.*

If $H \to X$ *admits a* Spin^c*-structure, then the set of isomorphism classes of* Spin^c*-structures is an affine space for* $H^2(X; \mathbb{Z})$*; the action of an element of* $H^2(X; \mathbb{Z})$ *is defined through the tensor product with the corresponding line bundle as in* (1.10).

2.2.1. *Conjugation of* Spin^c*-structures.* The conjugation operation introduced in section 2.1.3 extends to an involution on Spin^c-structures. We denote this map by $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$. It may happen that $\bar{\mathfrak{s}}$ is in fact isomorphic to $\mathfrak{s}$. If so, we make the following definition.

DEFINITION 1.5. A Spin^c-structure $\mathfrak{s}$ together with a choice of isomorphism, $J\colon \bar{\mathfrak{s}} \to \mathfrak{s}$, is called a *spin structure.*

We can think of J as a $\mathbb{C}$-anti-linear isomorphism $V \to V$ commuting with γ.

In any case there is a unique complex line bundle $L_{\mathfrak{s}}$ so that $\mathfrak{s} = \bar{\mathfrak{s}} \otimes L_{\mathfrak{s}}$. This $L_{\mathfrak{s}}$ is called the *characteristic* line bundle of the Spin^c-structure $\mathfrak{s}$. The first Chern class of $L_{\mathfrak{s}}$ is an integral lift of $w_2(H)$, the second Stiefel-Whitney class of H. This will be defined below.

2.3. Spin^c**-structures via principal bundles and Stiefel-Whitney classes.** The definition of a Spin^c-structure can be rephrased in terms of principal bundles. Let $\gamma_0 : \mathbb{R}^n \to \mathrm{End}(\mathbb{C}^N)$ be a standard Clifford module. One can define a group $\mathrm{Spin}^c(n)$ as the set of pairs

$$(\tau, \sigma) \in \mathrm{SO}(n) \times \mathrm{U}(N)$$

for which the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tau} & \mathbb{R}^n \\ {\scriptstyle\gamma_0}\downarrow & & {\scriptstyle\gamma_0}\downarrow \\ \mathrm{End}(\mathbb{C}^N) & \xrightarrow{\mathrm{A}d(\sigma)} & \mathrm{End}(\mathbb{C}^N) \end{array},$$

where $\mathrm{A}d(\sigma)$ denotes the adjoint action of σ on $\mathrm{End}(\mathbb{C}^N)$. The homomorphism $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ given by τ is surjective because of the uniqueness of standard Clifford modules, and its kernel is the automorphism group of the module, which is S^1. So there is a short exact sequence

$$S^1 \xrightarrow{j} \mathrm{Spin}^c(n) \to \mathrm{SO}(n) . \tag{1.11}$$

To give a Spin^c-structure for an oriented vector bundle $H \to X$ with a metric is equivalent to giving a principal $\mathrm{Spin}^c(n)$-bundle $Q \to X$ together with an isomorphism of $Q/S^1 \to X$ with $\mathrm{Fr}(H) \to X$, the oriented orthonormal frame bundle.

In one direction, given a Spin^c-structure $\mathfrak{s} = (V, \gamma)$, one can construct $Q \to X$ by defining its fibers Q_x as sets of pairs $(t, s) \in \mathrm{Hom}(\mathbb{R}^n, H_x) \times \mathrm{Hom}(\mathbb{C}^N, V_x)$ which preserve both inner products and make the following diagram commutative:

$$\begin{array}{ccc} H_x & \xleftarrow[t]{} & \mathbb{R}^n \\ {\scriptstyle\gamma_x}\downarrow & & {\scriptstyle\gamma_0}\downarrow \\ \mathrm{End}(V_x) & \xleftarrow[\mathrm{A}d(s)]{} & \mathrm{End}(\mathbb{C}^N) \end{array}.$$

The map from Q to $\mathrm{Fr}(H)$ is given by $(t, s) \mapsto t$.

In the other direction, given Q, one can associate to it a vector bundle V via the homomorphism $\mathrm{Spin}^c(n) \to \mathrm{U}(N)$ provided by σ; and γ is then obtained from γ_0.

The proof of Proposition 1.4 can now be rephrased as follows. A principal G-bundle is a 1-cochain with values in $\mathcal{S}_G$, and the isomorphism class of the bundle corresponds to the cohomology class represented by this cochain.

The short exact sequence of sheaves

$$\mathcal{S}_{S^1} \to \mathcal{S}_{\mathrm{Spin}^c(n)} \to \mathcal{S}_{\mathrm{SO}(n)}$$

gives rise to an exact sequence in cohomology, which reads in part:

$$H^1(X;\mathcal{S}_{\mathrm{Spin}^c(n)}) \to H^1(X;\mathcal{S}_{\mathrm{SO}(n)}) \to H^2(X;\mathcal{S}_{S^1}) \ . \tag{1.12}$$

This is an exact sequence of pointed sets, the base point being the isomorphism class of the trivial bundle. Due to the nonabelian coefficient groups, the sequence does not extend in a naive way.

The isomorphism class of a principal $\mathrm{SO}(n)$-bundle $\mathrm{Fr}(H)$ gives an element of $H^1(X;\mathcal{S}_{\mathrm{SO}(n)})$. A Spin^c-structure exists if and only if this element is in the image of $H^1(X;\mathcal{S}_{\mathrm{Spin}^c(n)})$. If the $\{\phi_{**}\}$ in (1.6) define a cocycle, then this cocycle represents such a lift. By exactness, the existence of the lift is equivalent to the vanishing of the image $\mathfrak{o}(H) \in H^2(X,\mathcal{S}_{S^1})$ of the isomorphism class of $\mathrm{Fr}(H)$. This is represented by the cochain $\{\psi_{***}\}$ in (1.7).

Spin^c-structures on $\mathrm{Fr}(H)$ are cocycles with values in $\mathcal{S}_{\mathrm{Spin}^c(n)}$ lifting the $\mathcal{S}_{SO(n)}$-cocycle given by $\mathrm{Fr}(H)$. Any two differ by a 1-cochain with values in $\mathcal{S}_{S^1}$. The cohomology class of this cochain is the difference element $\delta(\mathfrak{s},\mathfrak{s}')$ in (1.9).

2.3.1. *Spin structures and principal bundles.* A Spin^c-structure, $\mathfrak{s}$, which admits an isomorphism $J\colon \overline{\mathfrak{s}} \to \mathfrak{s}$ gives rise to a spin structure. In terms of principal bundles the choice of J defines a reduction P of the structure group of the principal $\mathrm{Spin}^c(n)$ bundle, Q, defined by $\mathfrak{s}$, to the subgroup $\mathrm{Spin}(n) \subset \mathrm{Spin}^c(n)$ preserving J fiberwise. In a single fiber, the automorphisms of a standard Clifford module preserving complex conjugation consist of $\{\pm 1\} \subset S^1$. Thus $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$:

$$\mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \ . \tag{1.13}$$

We leave to the reader to check that this is the connected double cover. Thus P is a connected double covering of the frame bundle of H. In this way we recover the usual definition of a spin structure.

Applying the sheaf cohomology argument above shows that the obstruction to the existence of a spin structure on H is a well-defined element of $H^2(X;\mathbb{Z}_2)$.

DEFINITION 1.6. For an oriented $\mathbb{R}^n$ bundle $H \to X$, the *second Stiefel-Whitney class* $w_2(H) \in H^2(X;\mathbb{Z}_2)$ is the obstruction to the existence of a spin structure.

The reader familiar with other definitions of Stiefel-Whitney classes should have no trouble verifying the equivalence of this definition with their favorite one.

From the inclusion of the sequence (1.13) in (1.11), one sees that $\mathfrak{o}(H)$ is the image of $w_2(H)$ under the map $H^2(X;\mathbb{Z}_2) \to H^2(X;\mathcal{S}_{S^1})$. Alternatively, if we identify $H^2(X;\mathcal{S}_{S^1})$ with $H^3(X;\mathbb{Z})$ as above, $\mathfrak{o}(H)$ is the image of $w_2(H)$ under a map $H^2(X;\mathbb{Z}_2) \to H^3(X;\mathbb{Z})$ which we can identify as the connecting homomorphism for the exact sequence

$$\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2\ . \tag{1.14}$$

From the long exact sequence in cohomology we deduce:

PROPOSITION 1.7. *The obstruction $\mathfrak{o}(H)$ to the existence of a* Spin^c-*structure for $H \to X$ vanishes if and only if $w_2(H)$ has a lift to $H^2(X;\mathbb{Z})$.*

By a Spin^c-structure on a Riemannian manifold X, we mean a Spin^c-structure for its cotangent bundle. Although not every vector bundle of rank 4 on a 4-manifold has a Spin^c-stucture, the cotangent bundle always has one.

THEOREM 1.8 (Whitney). *Every oriented 4-manifold X admits a* Spin^c*-structure. That is, $w_2(X)$ is the reduction of an integer class.*

PROOF. We consider the commutative diagram

$$\begin{array}{ccccccc}
Ext(H_1(X;\mathbb{Z});\mathbb{Z}) & \xrightarrow{f} & H^2(X;\mathbb{Z}) & \xrightarrow{g} & Hom(H_2(X;\mathbb{Z});\mathbb{Z}) & \longrightarrow & 0 \\
{\scriptstyle r}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle r}\downarrow & & \\
Ext(H_1(X;\mathbb{Z});\mathbb{Z}_2) & \xrightarrow{f} & H^2(X;\mathbb{Z}_2) & \xrightarrow{g} & Hom(H_2(X;\mathbb{Z});\mathbb{Z}_2) & \longrightarrow & 0\ ,
\end{array}$$

where the vertical maps are given by reduction modulo 2 of the coefficients, and the rows arise from the universal coefficient theorem and are exact. We need to prove that there is a class $c \in H^2(X;\mathbb{Z})$ with $r(c) = w_2(X)$. Let $w = g(w_2(X))$, then $w(x) = \langle w_2(X), x\rangle$ for $x \in H_2(X;\mathbb{Z})$. The image of r in $Hom(H_2(X;\mathbb{Z});\mathbb{Z}_2)$ consists of the homomorphisms $H_2(X;\mathbb{Z}) \to \mathbb{Z}_2$ that annihilate the torsion subgroup of $H_2(X;\mathbb{Z})$. We claim that this is the case for w. Indeed, if $x \in H_2(X;\mathbb{Z})$ is a torsion class, then $x \cdot x = 0$, hence $w(x) \equiv x \cdot x \equiv 0 \mod 2$.

Thus we can find an element $\alpha \in H^2(X;\mathbb{Z})$ such that $\beta = r(\alpha)$ has the same image under g as w_2. By exactness, there is a $\gamma \in Ext(H_1(X;\mathbb{Z});\mathbb{Z}_2)$ with $f(\gamma) = \beta - w_2$, and since the restriction r is a surjection on $Ext(H_1(X;\mathbb{Z});\mathbb{Z})$, we can find $\delta \in Ext(H_1(X;\mathbb{Z});\mathbb{Z})$ with $(r \circ f)(\delta) = \beta - w_2$. Let $c = \alpha - f(\delta)$. Then $r(c) = \beta - (\beta - w_2) = w_2$, hence c is an integral class lifting $w_2(X)$. □

3. Dirac operators

Let X be a smooth manifold with a Riemannian metric g. We now seek to globalize the right-hand-side of (1.3). For that we will need a Spin^c-structure $\mathfrak{s} = (V, \gamma)$ for the cotangent bundle T^*X and a connection in V. The latter is used to replace the partial derivatives in (1.3) with covariant derivatives. The global version of (1.3) is

$$D_A = \sum_{i=1}^{n} \gamma(dx^i)\nabla_{A,\frac{\partial}{\partial x^i}}, \tag{1.15}$$

where ∇_A denotes the covariant derivative of the connection A. This expression can easily be seen to be independent of the coordinate system since it is the composition of two intrinsically defined operations. First we write the covariant derivative of A as a map

$$\nabla_A \colon \Gamma(V) \to \Gamma(T^*X \otimes V) \ .$$

Second, we have the linear map induced from Clifford multiplication

$$\Gamma(T^*X \otimes V) \to \Gamma(V) \ .$$

Their composition is a first order differential operator D_A . While this operator can be considered for an arbitrary connection A in V, in practice we shall restrict ourselves to so called Spin^c-connections satisfying additional properties given in following subsection. In this case D_A will be called the *Dirac operator* associated to the connection A for the Spin^c-structure $\mathfrak{s}$. The Spin^c-structure is clear from the context and is not made explicit in the notation.

3.1. Spin^c**-connections.** Consider again an arbitrary oriented vector bundle with inner product H with a Spin^c-structure $\mathfrak{s} = (V, \gamma)$. Given an orthogonal connection B in H, there are two natural constraints to impose on a connection A in V. We can require that it is compatible with the Hermitian metric on V. In addition, we can ask that Clifford multiplication is parallel in the sense that for all sections Φ of V and sections h of H we have

$$\nabla_A(\gamma(h)\Phi) = \gamma(\nabla_B h)\Phi + \gamma(h)\nabla_A\Phi \ . \tag{1.16}$$

Thus, if h is parallel, then Clifford multiplication by h commutes with ∇_A.

Definition 1.9. A Hermitian connection A satisfying equation (1.16) is called a Spin^c-*connection.*

The basic issues to address are existence and uniqueness of such connections.

LEMMA 1.10. *For every* Spin^c*-structure* $\mathfrak{s}$ *there are* Spin^c*-connections on* V*. Given any two such connections* A_1 *and* A_2 *there is a one-form* θ *so that*

$$\nabla_{A_2} - \nabla_{A_1} = i\theta \otimes \mathrm{Id}_V \ .$$

PROOF. To prove existence we can argue locally, and then patch together using a partition of unity.

First notice that we can locally give H a flat connection τ induced from an oriented orthogonal trivialization. We then choose a trivialization for V so that Clifford multiplication by a parallel section with respect to τ is given by a constant matrix. Then the product connection, A_0, with respect to this trivialization of V is compatible with τ. Let $\Theta = \nabla_B - \nabla_\tau$. We seek a skew-Hermitian $\mathrm{End}(V)$-valued one-form $\tilde{\Theta}$ so that $A_0 + \tilde{\Theta}$ is compatible with ∇_B in the sense of (1.16). Every such $\tilde{\Theta}$ gives us a local Spin^c-connection.

Taking the difference of the corresponding versions of equation (1.16), this implies that

$$\tilde{\Theta}\gamma(h) - \gamma(h)\tilde{\Theta} = \gamma(\Theta(h)) \ . \tag{1.17}$$

The isomorphism $\gamma\colon \Lambda^*(H) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}(V)$ makes $\mathrm{End}(V)$ into a $\mathbb{Z}$-graded vector space. Given $M \in \mathrm{End}(V)$ we write

$$M = \sum_{\nu=0}^{n} M_\nu$$

for the decomposition according to the grading. Notice that if $h_1, \ldots, h_n$ is an orthonormal basis for H, then

$$\sum_{i=1}^{n} \gamma(h_i) M \gamma(h_i) = \sum_{\nu=0}^{n} \big((-1)^{\nu+1}(n-2\nu)\big) M_\nu \ .$$

Thus equation (1.17) implies

$$\begin{aligned}\sum_{i=1}^{n} \gamma(h_i)\gamma(\Theta(h_i)) &= \sum_{i=1}^{n} (\gamma(h_i)\tilde{\Theta}\gamma(h_i) + \tilde{\Theta}) \\ &= \sum_{\nu=0}^{n} (n + (-1)^{\nu+1}(n-2\nu))\tilde{\Theta}_\nu \ .\end{aligned}$$

The left-hand side of the equation lies in $\gamma(\Lambda^2)$ hence

$$\tilde{\Theta}_2 = \frac{1}{4} \sum_{i=1}^{n} \gamma(h_i)\gamma(\Theta(h_i))$$

while $\tilde{\Theta}_\nu = 0$ for $\nu \neq 0, 2, n$.

Thus, the equation for $\tilde{\Theta}$ has solutions. The most general solution is

$$\tilde{\Theta} = i\theta \operatorname{Id}_V + \frac{1}{4}\sum_{i=1}^{n} \gamma(h_i)\gamma(\Theta(h_i)) \ , \tag{1.18}$$

where θ is an arbitrary real one-form on X.

Having found the most general $\tilde{\Theta}$ also settles the (non-)uniqueness question. □

3.1.1. *Conjugation and Spin connections.* Suppose we have a Spin^c-structure $\mathfrak{s}$ which comes from a spin structure, so that a $\mathbb{C}$-anti-linear isomorphism $J\colon V \to V$ is fixed.

DEFINITION 1.11. A *spin connection* is a Spin^c-connection that commutes with J.

Unlike Spin^c-connections, spin connections are unique. The analog of Lemma 1.10 is simply:

LEMMA 1.12. *For every spin structure there is a unique spin connection on V.*

PROOF. By Lemma 1.10, V admits Spin^c-connections A. But, given J, we can choose the A_0 in the proof of Lemma 1.10 to commute with J. Then $\tilde{\Theta}_2$ also commutes with J. However, $\tilde{\Theta}_0 = i\theta \operatorname{Id}_V$ only commutes with J if $\theta = 0$. □

A Spin^c-connection A in V induces a connection $\hat{A}$ in the characteristic line bundle $L_{\mathfrak{s}}$. The trivialization of V from the proof of Lemma 1.10 induces a trivialization of $L_{\mathfrak{s}}$ in which the connection one-form of $\hat{A}$ is $2i\theta$.

3.2. The square of a Dirac operator. We will compare the square of the Dirac operator with the covariant Laplacian, $\nabla_A^*\nabla_A$, and the curvature of $\hat{A}$. The main result is the so called Weitzenböck formula for the Dirac operator.

THEOREM 1.13. *Let $\mathfrak{s}$ be a* Spin^c*-structure on an oriented Riemannian manifold X, and A a* Spin^c*-connection. Then the square of the associated Dirac operator satisfies the identity*

$$D_A^2 = \nabla_A^*\nabla_A + \frac{1}{4}s_g + \frac{1}{2}\gamma(F_{\hat{A}}) \ . \tag{1.19}$$

PROOF. In the above formula $F_{\hat{A}}$ is an imaginary two-form and $\gamma(F_{\hat{A}})$ denotes the the extension of Clifford multiplication to forms.

We will do the calculation in two steps. First recall that the curvature of A is a two-form with values in $\mathrm{End}(V)$ given by the expression:

$$F_A(X,Y) = [\nabla_{A,X}, \nabla_{A,Y}] - \nabla_{A,[X,Y]} \ .$$

Locally

$$F_A = \frac{1}{2}\sum_{i,j=1}^{n} F_{ij} e_i \wedge e_j \ ,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame for T^*X, and the F_{ij} are local sections of $\mathrm{End}(V)$ with $F_{ij} = -F_{ji}$.

Using Clifford multiplication we define an endomorphism $\gamma(F_A)$ of V as the image of F_A under the composition

$$\Lambda^2 \otimes \mathrm{End}(V) \xrightarrow{\gamma\otimes\mathrm{Id}} \mathrm{End}(V) \otimes \mathrm{End}(V) \longrightarrow \mathrm{End}(V) \ .$$

The second map is given by composition of endomorphisms. Thus locally we have:

$$\gamma(F_A)(\Phi) = \frac{1}{2}\sum_{i,j} \gamma(e_i \wedge e_j)(F_{ij}(\Phi)) \ .$$

Using the metrics on V and T^*X, the covariant derivative

$$\nabla_A \colon \Gamma(V) \longrightarrow \Gamma(V \otimes T^*X)$$

has a formal adjoint

$$\nabla_A^* \colon \Gamma(V \otimes T^*X) \to \Gamma(V) \ .$$

Here is the Weitzenböck decomposition of D_A^2, expressing it as a sum of the connection Laplacian and a curvature term.

LEMMA 1.14.

$$D_A^2 \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{2}\gamma(F_A)\Phi \ . \tag{1.20}$$

PROOF. Fix a point $x \in X$ and choose an orthonormal frame $\{e_1, \dots, e_n\}$ for T^*X with $(\nabla^{\mathrm{LC}} e_i)|_x = 0$. Writing ∇_i for the covariant derivative with respect to A in the e_i-direction, the definition of the Dirac operator gives:

$$\begin{aligned}
(D_A^2\Phi)|_x =&\big(\sum_{i,j}\gamma(e_i)(\nabla_i(\gamma(e_j)(\nabla_j\Phi)))\big)|_x\\
=&\big(\sum_i(-\nabla_i\nabla_i)(\Phi)+\sum_{i<j}\gamma(e_i\wedge e_j)(\nabla_i\nabla_j-\nabla_j\nabla_i)(\Phi)\big)|_x\\
=&\big(\nabla_A^*\nabla_A\Phi+\frac{1}{2}\sum_{i,j}\gamma(e_i\wedge e_j)F_{ij}(\Phi)\big)|_x\\
=&\big(\nabla_A^*\nabla_A\Phi+\frac{1}{2}\gamma(F_A)\Phi\big)|_x\ .
\end{aligned}$$

□

It remains to compute $\gamma(F_A)$ when A is Spin^c-connection. From equation (1.18) we saw that in a trivialization the connection one-form of a Spin^c-connection is $i\theta\,\mathrm{Id}_V+\frac{1}{4}\sum_{i=1}^n\gamma(e_i)(\Theta^{\mathrm{LC}}(e_i))$, where Θ^{LC} is the connection one-form of the Levi-Civita connection. Thus the curvature is

$$F_A=id\theta\,\mathrm{Id}_V+\frac{1}{4}\sum_{i=1}^n\gamma(e_i)\gamma(R^{\mathrm{LC}}(e_i))$$

and writing the Riemannian curvature locally as

$$R^{\mathrm{LC}}(e_i)=\frac{1}{2}\sum_{j,k,l=1}^n(R_{ijkl}e_j\wedge e_k)\otimes e_l$$

we have

$$\begin{aligned}
\frac{1}{2}\gamma(F_A)&=\frac{1}{2}i\gamma(d\theta)+\frac{1}{8}\sum_{i,j,k,l=1}^n\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)R_{ijkl}\\
&=\frac{1}{2}\gamma(F_{\hat{A}})+\frac{1}{4}s_g\ ,
\end{aligned}$$

where we have used the Bianchi identity to cancel terms not having two pairs of indices the same. □

3.3. The indices of Dirac operators. The Dirac operator is the prototypical example of a first order elliptic operator.

Let $P\colon\Gamma(E)\to\Gamma(F)$ be a first order operator between vector bundles E and F over the closed manifold X. Its symbol

$$\sigma(P)\colon T^*X\to\mathrm{Hom}(E,F)$$

is defined as follows. Given $\xi\in T_x^*X$ and $e\in E_x$, choose a function f with $f(x)=0$ and $df|_x=\xi$, and an extension $\tilde{e}$ of e to a section of E.

Then

$$\sigma(P)(\xi)e = P(f\tilde{e})|_x \ .$$

It is a standard exercise to check that this independent of the choices of function and extension. The operator P is called *elliptic* if its symbol is invertible for all non-zero ξ.

The symbol of a Dirac operator D_A is Clifford multiplication, and hence D_A is elliptic.

A basic result about elliptic operators is that their kernels are finite-dimensional. If an operator P is elliptic then so is its formal adjoint, P^*, and hence we can define the *index* of P to be

$$\mathrm{ind}(P) = \dim \mathrm{Ker}(P) - \dim \mathrm{Ker}(P^*) \ .$$

We leave it to the reader to check that the Dirac operator D_A is formally self-adjoint.

Thus the index of D_A is always zero. In spite of this there is an interesting index problem associated with Dirac operators. Recall that we saw in dimension four that the volume form acts as an involution on a Clifford module. This is always true in dimensions $4k$. We can then split $V = V_+ \oplus V_-$, with the convention that the volume form acts by $(-1)^k$ on V_+ and by $-(-1)^k$ on V_-. The convention is chosen so that self-dual forms act trivially on V_- and anti-self-dual forms act trivally on V_+. Since the volume form is parallel, Clifford multiplication by it commutes with the covariant derivative of any Spin^c-connection. On the other hand, Clifford multiplication by one-forms interchanges the summands. Thus the Dirac operator D_A decomposes into two operators

$$D_A^{\pm} \colon \Gamma(V_{\pm}) \to \Gamma(V_{\mp}) \ .$$

Now we have $(D_A^{\pm})^* = D_A^{\mp}$. Therefore $\mathrm{ind}(D_A^+) = -\,\mathrm{ind}(D_A^-)$, but there is no reason that either side should vanish.

In the case that our Spin^c-structure comes from a spin structure, the index of the Dirac operator is given by the Atiyah-Singer index theorem in terms of the $\hat{A}$-genus:

$$\mathrm{ind}_{\mathbb{C}}\, D^+ = \langle \hat{A}(M), [M] \rangle \ . \tag{1.21}$$

Using $\hat{A}(M) = 1 - \frac{1}{24}p_1(M) + \ldots$ and the Hirzebruch signature theorem in dimension four, $\langle p_1(M), [M]\rangle = 3\sigma(M)$, we see that when M is a 4-manifold, then

$$\mathrm{ind}_{\mathbb{C}}\, D^+ = -\frac{1}{8}\sigma(M) \ . \tag{1.22}$$

One can deduce from this formula Rokhlin's theorem which shows that the spin condition imposes a non-trivial restriction on the intersection form of a manifold. Rokhlin's original proof was homotopy theoretic in nature.

THEOREM 1.15 (Rokhlin). *The signature of a smooth closed oriented spin* 4*-manifold is divisible by* 16.

PROOF. Let X be a closed spin 4-manifold. For each choice of spin structure (V, γ, J) on X, the Dirac operator $D^+ \colon \Gamma(V_+) \to \Gamma(V_-)$. The isomorphism J commutes with Clifford multiplication and is parallel, so it commutes with D^+. We have $J^2 = -1$, and so J gives the kernel and cokernel the structure of quaternionic vector spaces. Thus $\mathrm{ind}_{\mathbb{C}} D^+$ must be even. □

The generalization of the index formula (1.21) to Spin^c-Dirac operators D_A^+ is the following:

$$ind_{\mathbb{C}} D_A^+ = \langle \hat{A}(X) e^{\frac{1}{2} c_1(L_{\mathfrak{s}})}, [X] \rangle \ , \tag{1.23}$$

where $L_{\mathfrak{s}}$ is the characteristic line bundle of the Spin^c-structure. In the 4-dimensional case we have $c_1(L_{\mathfrak{s}}) = c_1(V_+)$, and

$$ind_{\mathbb{C}} D_A^+ = -\frac{1}{8}\sigma(X) + \frac{1}{2} c_1^2(V_+) \ , \tag{1.24}$$

where we now think of $c_1^2(V_+)$ as having been evaluated on the fundamental class of X.

The Weitzenböck formula simplifies greatly in the spin case, and this has very strong consequences:

$$D^- D^+ = \nabla^* \nabla + \frac{1}{4} s_g \ . \tag{1.25}$$

Of course, the same formula also holds for $D^+ D^-$. From this we deduce the following result:

THEOREM 1.16 (Lichnerowicz). *If a closed oriented spin* 4*-manifold admits a metric of positive scalar curvature, then its signature vanishes.*

PROOF. Let g be a metric of positive scalar curvature on the closed spin manifold X, and let $\phi \in \Gamma(V_+)$ be in the kernel of the Dirac operator D^+. Applying equation (1.25) to ϕ and taking the L^2 inner product of the result with ϕ we obtain

$$0 = \int_X |\nabla \phi|^2 dvol_g + \frac{1}{4} \int_X s_g |\phi|^2 dvol_g \ .$$

This shows that ϕ vanishes identically. Similarly, the kernel of D^-, equivalently the cokernel of D^+, is trivial. This implies that $indD^+ = 0$, and the conclusion follows from the index formula (1.22). $\square$

REMARK 1.17. Throughout this book, such statements about topological restrictions on manifolds admitting metrics of positive scalar curvature will appear. This condition could be phrased in various other ways, either seemingly stronger, or seemingly weaker. However, the following natural conditions on a closed manifold X are all equivalent:

(1) X admits a metric of constant positive scalar curvature,
(2) X admits a metric of positive scalar curvature,
(3) X admits a metric of non-negative scalar curvature which is not identically zero.

It is clear that the conditions are successively weaker. But the last condition implies the first, because of the positive solution to the Yamabe problem due to Schoen [**28**].

3.4. The Weitzenböck formula for positive spinors. In the four-dimensional case the extrinsic curvature term in the Weitzenböck decomposition has a special property. Since the anti-self-dual two-forms act trivially on V_+, equation 1.19 becomes:

$$D_A^- D_A^+ \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{4} s_g \Phi + \frac{1}{2} \gamma(F_{\hat{A}}^+) \Phi \ . \tag{1.26}$$

This formula is at the root of many of the important features of the monopole equations. The analogous formula for negative spinors also holds.

Notes:

1. The term *spinors* was introduced by Dirac, though the objects themselves were first considered by É. Cartan.

2. For background on sheaf theory and obstructions, see [**2, 25, 36**].

3. The Weitzenböck formula for the Dirac operator goes back at least to Weitzenböck, who however attributes the curvature calculation to Ricci and Levi-Civita. It was rediscovered by Schrödinger and many others.

4. Our proof of Whitney's theorem is from Bohr [**4**].