Geometry of manifolds I
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(1) Topological spaces; open and closed sets; Hausdorffness; continuity.
(2) Metric spaces, and their induced topology. This topology is always Hausdorff.
(3) Euclidean space $\mathbb{R}^n$ with its metric topology. By considering balls with rational radii centered at points with rational coordinates, one finds a countable collection of open sets such that all open sets are suitable unions of these. This is abstracted into the notion of a countable basis for the topology.
(4) **Topological manifolds** are locally Euclidean spaces that are Hausdorff and have a countable basis for their topology.
(5) A **differentiable manifold** is a topological manifold together with an atlas whose transition maps are C^r diffeomorphisms for some $r \geq 1$. A **differentiable structure** is an equivalence class of atlases, equivalently a maximal atlas.
(6) Every maximal C^r atlas with $r \geq 1$ contains a C^∞ atlas. Because of this fact (which we do not prove), we will restrict ourselves to C^∞ manifolds throughout. So the word differentiable will usually mean C^∞.
(7) For a differentiable manifold we define differentiability of functions $f\colon M \longrightarrow \mathbb{R}$.

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(8) Differentiable functions and maps; diffeomorphisms vs. homeomorphisms.
(9) Dimensions of manifolds and smooth invariance of domain.
(10) Examples of differentiable manifolds and their dimensions: Euclidean spaces $\mathbb{R}^n$, spheres S^n, tori T^n, $GL(n, \mathbb{R})$,... An open subset of a manifold is a manifold (of the same dimension); products of manifolds are manifolds (and the dimensions add up).
(11) Every manifold M is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering $\{U_i\}_{i\in I}$ of M, there is an atlas $\{(V_k, \varphi_k)\}$ such that the covering by the V_k is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball B_3 of radius 3 for all k and the open sets $W_k = \varphi_k^{-1}(B_1)$ cover M.

Proof. We prove first that there is a sequence G_i, $i = 1, 2, \ldots$ of open sets with compact closures, such that the G_i cover M and satisfy

$$\overline{G_i} \subset G_{i+1}$$

for all i. To this end let A_i, $i = 1, 2, \ldots$ be a countable basis of the topology consisting of open sets with compact closures. (This exists because M is second countable, Hausdorff and locally compact, see the homework assignement.) Set $G_1 = A_1$. Suppose inductively that we have defined

$$G_k = A_1 \cup \ldots \cup A_{j_k} \ .$$

Then let j_{k+1} be the smallest integer greater than j_k with the property that

$$\overline{G_k} \subset A_1 \cup \ldots \cup A_{j_{k+1}} \ ,$$

and define

$$G_{k+1} = A_1 \cup \ldots \cup A_{j_{k+1}} \ .$$

This defines the sequence G_k as desired.

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We continue the proof we began last time.

Let $\{U_i\}_{i\in I}$ be an arbitrary open covering of M. For every $x \in M$ we can find a chart (V_x, φ_x) at x with V_x contained in one of the U_i and such that $\varphi_x(V_x) = B_3$. Let $W_x = \varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_k} \setminus G_{k-1}$ by finitely many such W_{x_j} such that at the same time the corresponding V_{x_j} are contained in the open set $G_{k+1} \setminus \overline{G_{k-2}}$. Taking all these V_{x_j} as i ranges over the positive integers we obtain the desired atlas. $\square$

(12) We construct smooth bump functions on $\mathbb{R}^n$ and transfer them to differentiable manifolds via charts. This allows us to construct various kinds of differentiable functions with special properties.
(13) Every open covering of a differentiable manifold admits a subordinate differentiable **partition of unity**. This follows from paracompactness and the existence of smooth bump functions.
(14) The **tangent bundle** TM of a differentiable manifold M of dimension n is itself a manifold of dimension $2n$.

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(15) The projection $\pi\colon TM \longrightarrow M$ is a differentiable map. The preimage $T_xM = \pi^{-1}(x)$ of any $x \in M$ carries a natural vector space structure. We call this the **tangent space** of M at x.
(16) For any differentiable map $f\colon M \longrightarrow N$, we define the **derivative** $Df\colon TM \longrightarrow TN$. This restricts to every tangent space T_xM as a linear map D_xf to $T_{f(x)}N$. This is the derivative of f at $x \in M$.
(17) Differentiable **vector bundles** over manifolds; see [1] Section 3.3. Local vs. global triviality; isomorphisms of bundles. The tangent bundle as a vector bundle. Sections of vector bundles.

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(18) The space $\Gamma(E)$ of smooth sections of a vector bundle $\pi\colon E \longrightarrow B$ is a vector space over $\mathbb{R}$ with the operations of addition and scalar multiplication defined point-wise. In the same way, it is also a module over the ring of smooth functions on B.
(19) For every $v \in E$ there exists an element $s \in \Gamma(E)$ with $s(\pi(v)) = v$.
(20) If both the rank of a vector bundle and the dimension of the base manifold are positive, then $\Gamma(E)$ is infinite-dimensional. However, if B is compact, then $\Gamma(E)$ always contains a finite-dimensional subspace V such that the evaluation map

$$\begin{aligned} ev\colon B \times V &\longrightarrow E \\ (x, s) &\longmapsto s(x) \end{aligned}$$

is surjective.

(21) A **metric** on a vector bundle is a fibre-wise positive definite scalar product that depends smoothly on the fibre. Using a partition of unity subordinate to a covering of B by trivializing open sets for E, we show that every vector bundle admits a metric.

3 November 2006

(22) A **curve** in a differentiable manifold M is a map c from $\mathbb{R}$ or from some subinterval of $\mathbb{R}$ to M. If such a map is differentiable at t, then its derivative $D_t c$ applied to the tangent vector $\frac{\partial}{\partial t}$ of $\mathbb{R}$ gives a vector in $T_{c(t)}M$ denoted by $\dot{c}(t)$ and called the **velocity vector** of c at t (or at $c(t)$). Every tangent vector to M can be realized as a velocity vector of a suitable curve.

(23) A metric on the vector bundle TM is called a **Riemannian metric** on M. As every vector bundle admits metrics, every differentiable manifold admits Riemannian metrics. Once such a metric g is chosen, we can define the length of any (piece-wise) C^1 curve c defined on a compact interval $[a, b]$ by

$$l(c) = \int_a^b \sqrt{g(\dot{c}(t), \dot{c}(t))}dt \ .$$

Assuming that M is connected, we then obtain a function

$$d\colon M \times M \longrightarrow \mathbb{R}$$

by letting $d(x, y)$ be the infimum of the lengths of all piece-wise C^1 curves starting at x and ending at y.

Theorem 1. *The function d defines the structure of a metric space on M. The metric topology of the metric space (M, d) coincides with the manifold topology of M.*

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(24) Completion of the proof of Theorem 1.

(25) Integration of **vector fields**; **local** and **global flows**; **completeness**, see [1] Section 4.1.

10 November 2006

(26) Every vector field with compact support is complete.

(27) Vector fields as **derivations** and the **Lie derivative**.

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(28) All derivations on the algebra $C^\infty(M)$ are vector fields.

(29) **Commutators** and the Lie algebra structure on the space of vector fields, see [1] Section 4.2.

17 November 2006

(30) A **differential form** of degree k on a smooth manifold M is a map

$$\omega\colon \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \longrightarrow C^\infty(M)$$

of $C^\infty(M)$-modules, in other words, it is function-linear in all k arguments. In addition, it is required to satisfy the following condition:

(1) $$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = sign(\sigma)\omega(X_1, \ldots, X_k)$$

for all permutations $\sigma \in \Sigma_k$.

(31) We have the following:

Lemma 2. *If ω is a differential form, then the value of the function $\omega(X_1, \ldots, X_k)$ at a point $p \in M$ depends on the vector fields X_i only through their values $X_i(p)$ at the point p.*

This means that ω has a value $\omega(p)$ at p, which is a k-multilinear map

$$\omega(p)\colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$$

defined on $(X_1(p), \ldots, X_k(p))$ by extending the $X_i(p)$ to global vector fields, evaluating ω on these vector fields, and then evaluating the resulting function at p. (This multilinear map of course inherits property (1).)

(32) We build a universal model for multilinear maps, first for vector spaces (like T_pM), and then for vector bundles (like TM). This will allow us to interpret differential forms as sections of suitable vector bundles, so that $\omega(p)$ will be simply the value of the section ω at p. The universal model for bilinear maps on $V \times W$ is given by the **tensor product**:

$$\otimes\colon V \times W \longrightarrow V \otimes W\ .$$

Iterating this we obtain tensor products of k vector spaces which have the universal property for k-linear maps. The **tensor algebra** and of a vector space V is the direct sum of the tensor products $T^k(V)$ of k copies of V, for $k = 0, 1, 2, \ldots$ endowed with the natural mutiplication given by the tensor product. Here $T^0(V)$ is just the ground field, and $T^1(V)$ is V itself. (See [1] Section 7.2.).

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(33) The **exterior algebra** of a vector space over a field of characteristic $\neq 2$. (See [1] Section 7.2.).

24 November 2006

(34) Multilinear algebra constructions applied to vector bundles.

28 November 2006

(35) **Differential forms** as sections of exterior powers of the cotangent bundle.

(36) The **exterior derivative**; existence and uniqueness.

1 December 2006

(37) **Pullback** of forms, contraction and the **Lie derivative** on forms, Cartan's formula

$$L_X = d \circ i_X + i_X \circ d \,. \tag{2}$$

5 December 2006

(38) **Orientability** and **orientations** on vector bundles.
(39) Orientability and orientations on manifolds via the tangent bundle.
(40) Existence and uniqueness of the **integral** of n-forms with compact support on oriented n-manifolds. (See [1] Section 8.2.)

8 December 2006

(41) **Manifolds with boundary**.
(42) Orientations of manifolds with boundary and the induced orientation on the boundary.
(43) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \,.$$

(See [1] Section 8.2.)

12 December 2006

(44) **Closed** and **exact** k-forms; the de Rham complex and its cohomology, called the **de Rham cohomology** $H^k_{dR}(M)$ of a differentiable manifold M.
(45) The forms with compact support form a subcomplex of the de Rham complex. Its cohomology is called the (de Rham) cohomology with compact support and denoted $H^k_c(M)$. For compact manifolds this is of course the same as the ordinary de Rham cohomology defined above.
(46) For any oriented n-dimensional manifold M without boundary, the integral gives a well-defined surjective linear map:

$$\int_M \colon H^n_c(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega \,.$$

(47) Any differentiable map $f\colon M \longrightarrow N$ induces a map on de Rham cohomology

$$f^*\colon H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

defined by pulling back closed forms. (Recall that on forms the pullback commutes with exterior differentiation.)
(48) The **Poincaré lemma**: If $i\colon M \longrightarrow M \times \mathbb{R}$ is the inclusion of M as $M \times \{0\}$ and $\pi\colon M \times \mathbb{R} \longrightarrow M$ is the projection, then i^* and π^* are inverses of each other on cohomology. Thus M and $M \times \mathbb{R}$ have isomorphic de Rham cohomology.

(49) As consequences of the Poincaré lemma we have in particular a complete calculation of the de Rham cohomology of $\mathbb{R}^n$ by induction on n, and the statement that on any manifold every closed form is locally exact.

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(50) The **Poincaré lemma** for de Rham cohomology with compact supports: $H^k_c(M)$ and $H^{k+1}_c(M \times \mathbb{R})$ are isomorphic for all all $k \geq 0$.
(51) By induction on n we find the cohomology of $\mathbb{R}^n$ with compact supports. It is trivial except in degree n, where the integration map gives an isomorphism with $\mathbb{R}$. We know from (46) that the integration map is well-defined, linear and surjective. Now injectivity follows because we know from the Poincaré lemma with compact supports that $H^n_c(\mathbb{R}^n) = \mathbb{R}$.
(52) Decomposing S^n with $n \geq 1$ into the union of two open sets diffeomorphic to $\mathbb{R}^n$, and applying both versions of the Poincaré lemma, for arbitrary forms and for those with compact supports, we calculate the de Rham cohomology of S^n, and find that it is isomorphic to $\mathbb{R}$ in degrees 0 and n, and trivial otherwise.

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(53) For any closed oriented n-dimensional manifold M without boundary, we have $H^n_{dR}(M) = \mathbb{R}$.
(54) Using the above, we prove Moser's result on **isotopy of volume forms**: if M is closed and oriented, then any two volume forms with the same total volume are diffeomorphic to each other by a diffeomorphism isotopic to the identity.

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(55) In the proof of Moser's theorem above we integrated a time-dependent vector field to construct a family of diffeomorphisms. This can be understood by translating a time-dependent vector field on M into a time-independent one on $M \times \mathbb{R}$. The flow on $M \times \mathbb{R}$ then produces the desired family φ_t of diffeomorphisms on M, which do not, however, form a flow, because in general $\varphi_t \circ \varphi_s \neq \varphi_{t+s}$.
(56) As a further illustration of the theory of differential forms, we discuss the **Calabi invariant** of area-preserving diffeomorphisms. Consider a volume form ω on the two-disk D^2. By the Poincaré lemma it is exact, so we can choose a one-form λ such that $\omega = -d\lambda$. Let

$$Diff_c(D^2, \omega) = \{\varphi \in Diff_c(D^2) \mid \varphi^*\omega = \omega\}$$

be the group of diffeomorphisms of D^2 preserving ω and having compact support in the interior of D^2. We define the Calabi invariant of a volume-preserving diffeomorphism by

$$\begin{aligned} Cal\colon Diff_c(D^2, \omega) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \int_{D^2} \varphi^*(\lambda) \wedge \lambda \ . \end{aligned}$$

Note that because φ has compact support, so does the integrand $\varphi^*(\lambda) \wedge \lambda$, and thus the integral makes sense.
(57) The Calabi invariant has the following properties:
- Cal is well-defined, independently of the choice of λ,
- it is a homomorphism of groups (addition defining the group structure on $\mathbb{R}$), and

- it is surjective.

9 January 2007

(58) We now begin the discussion of **connections** and **curvature** on vector bundles.

Let $E \to B$ be a differentiable vector bundle of rank k over a smooth manifold B of dimension n.

Definition 3. A **connection** on E is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(E) \longrightarrow \Omega^1(E) \tag{3}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s) \tag{4}$$

for all $f \in C^\infty(B)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*B \otimes E)$ is the space of 1-forms on B with values in E. One can evaluate the 1-form on a vector field X to obtain

$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E)\ . \tag{5}$$

(59) We prove the following fundamental properties of connections:

- A connection ∇ does not increase the support of sections, i. e. if $s \in \Gamma(E)$ vanishes on some open set $U \subset B$, then so does $\nabla(s)$.
- The value of $\nabla(s)$ at a point $p \in B$ depends only on the restriction of s to an orbitrarily small open neighbourhood of p. (In other words, ∇ is a differential operator, and $\nabla(s)(p)$ depends only on the germ of s at p.)
- If ∇_1 and ∇_2 are connections, then so is $t\nabla_1 + (1-t)\nabla_2$ for all $t \in [0,1]$.
- If ∇_1 and ∇_2 are connections, then $\nabla_1 - \nabla_2 \in \Omega^1(End(E)) = \Gamma(T^*B \otimes E^* \otimes E)$.

(60) Using these properties and a partition of unity subordinate to a covering of B by open sets over which the restriction of E is trivial, we prove:

Proposition 4. *Every vector bundle E admits connections. The space of all connections on E is an affine space for the space $\Omega^1(End(E))$ of 1-forms on B with values in $End(E)$.*

(61) Next we extend the differential operator given by a connection ∇ to bundle-valued forms of higher degree.

Lemma 5. *For every connection ∇ on $E \to B$ there is a unique $\mathbb{R}$-linear map*

$$\bar\nabla\colon \Omega^l(E) \longrightarrow \Omega^{l+1}(E)$$

which satisfies

$$\bar\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^l \omega \wedge \nabla(s) \tag{6}$$

for all $\omega \in \Omega^l(B)$ and $s \in \Gamma(E)$. Moreover, this operator satisfies

$$\bar\nabla(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f\bar\nabla(\omega \otimes s) \tag{7}$$

for all smooth functions f on B.

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(62) Consider the operator $\bar{\nabla} \circ \nabla \colon \Omega^0(E) \longrightarrow \Omega^2(E)$ associated with a connection ∇ on E. It turns out that this is linear over $C^\infty(B)$, and is therefore given by an element $F^\nabla \in \Omega^2(End(E))$. This is called the **curvature** of ∇.

(63) A (local) **frame** for E is a set of smooth sections $s_1, \ldots, s_k$ defined over some open set $U \subset B$, whose values are linearly independent at every point $p \in U$.

Thus a set of k local smooth sections $s_1, \ldots, s_k$ is a frame if and only if $s_1(p), \ldots, s_k(p)$ is a basis of $E_p = \pi^{-1}(p)$ for every $p \in U$. Therefore a frame defined over U defines a trivialization of $E|_U$, and, conversely, every such trivialization

$$\psi \colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

defines a local frame by setting $s_i(p) = \psi^{-1}(p, e_i)$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$.

(64) Fix a local frame $s_1, \ldots, s_k$ for the restriction of E to a trivialising open set in M. This choice determines a connection ∇_0 defined by the requirement $\nabla_0(s_i) = 0$ for all i. Every other connection ∇ differs from ∇_0 by the addition of a 1-form with values in $End(E)$. However, the given trivialization of E induces a trivialization of $End(E)$, and so a 1-form with values in $End(E)$ is nothing but a $k \times k$ matrix of ordinary 1-forms. Thus ∇ can be expressed by the matrix $\omega = (\omega_{ij})$ of 1-forms given by

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

(65) From the definition of the curvature we calculate

$$F^\nabla(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} \ .$$

We can write this briefly as $\Omega = d\omega - \omega \wedge \omega$, where the wedge product on the right-hand-side includes matrix multiplication, and is therefore not necessarily trivial unless $k = 1$.

(66) Similarly we compute $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$. This is the **Bianchi identity**.

(67) Suppose we have another frame $s_1', \ldots, s_k'$ on the same domain of definition as the original frame. Let ω' and Ω' denote the connection and curvature matrices of ∇ with respect to this new frame. If

$$s_i' = \sum_{i=1}^{k} g_{ij} s_j \ ,$$

we find the following relations: $\omega' = dg\, g^{-1} + g\omega g^{-1}$ and $\Omega' = g\Omega g^{-1}$, where $g = (g_{ij})$. The change of basis g is called a **gauge transformation**, and these formulae show how connection and curvature matrices behave under gauge transformations. The curvature matrix Ω is more invariant than the connection matrix ω.

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(68) Recall that with respect to a framing $s_1, \ldots, s_k$ of E a connection ∇ is expressed by a matrix (ω_{ij}) of one-forms. If we choose a chart for the base manifold B with local coordinates $y_1, \ldots, y_n$, then in the domain of this chart every one-form can be expressed uniquely as a linear combination of the dy_i. In particular, there are smooth functions ω_{ij}^α on the domain of the chart such that

$$\omega_{ij} = \sum_{\alpha=1}^{n} \omega_{ij}^\alpha dy_\alpha \ . \tag{8}$$

Denoting the vector fields $\frac{\partial}{\partial y_\alpha}$ by ∂_α, we find the following:

$$\nabla_{\partial_\alpha} s_i = \langle \partial_\alpha, \nabla s_i \rangle = \sum_{j=1}^{k} \langle \partial_\alpha, \omega_{ij} \rangle s_j = \sum_{j=1}^{k} \omega_{ij}^\alpha s_j \ .$$

More generally, if

$$s = \sum_{i=1}^{k} x_i s_i \ ,$$

then

$$\nabla_{\partial_\alpha} s = \sum_{j=1}^{k} (\frac{\partial x_j}{\partial y_\alpha} + \sum_{i=1}^{k} x_i \omega_{ij}^\alpha) s_j \ .$$

Writing A^α for the matrix (ω_{ij}^α) of functions we see that the operator ∇_{∂_α}, which we abbreviate to ∇_α, has the form $\nabla_\alpha = \partial_\alpha + A^\alpha$.

We can now characterize flatness of ∇ through the condition that the directional derivatives ∇_α commute:

Proposition 6. *The connection ∇ is flat if and only if $[\nabla_\alpha, \nabla_\beta] = 0$ for every local coordinate system $y_1, \ldots, y_n$ on the base manifold M.*

(69) If $E \to B$ is a vector bundle with a connection ∇, we say that a section $s \in \Gamma(E)$ is **parallel** with respect to ∇ if $\nabla s = 0$. In the special case that ∇ is the connection given by some trivialization, a section is parallel if and only if it is constant in the given trivialization. Thus parallel sections should be thought of as the analogs of constant sections for nontrivial bundles.

(70) We will want to prove the following:

Proposition 7. *Let $\pi \colon E \to B$ be a vector bundle with a connection ∇, and $c \colon [0,1] \to B$ a smooth curve in the base space. Then for every $v \in \pi^{-1}(c(0))$ there is a unique smooth curve $\tilde{c} \colon [0,1] \to E$ with $\pi \circ \tilde{c} = c$, $\tilde{c}(0) = v$ and $\nabla_{\dot{c}} s = 0$, where s sends $c(t)$ to $\tilde{c}(t)$. Moreover, the map $v \mapsto \tilde{c}(1)$ defines a linear map of vector spaces $\pi^{-1}(c(0)) \to \pi^{-1}(c(1))$.*

(71) In Proposition 7 the condition $\nabla_{\dot{c}} s = 0$ makes sense although s is not a section over all of B because the covariant derivative is only considered in the direction of c, where s is defined.

The Proposition follows from the existence and uniqueness of the solutions of systems of linear ordinary differential equations with given initial conditions, together with the linear dependence of the solutions on the initial values.

Definition 8. The linear map

$$\begin{aligned} P_t \colon E_{c(0)} &\longrightarrow E_{c(t)} \\ v &\longmapsto \tilde{c}(t) \end{aligned}$$

is called the **parallel transport** along c. It is an isomorphism of vector spaces.

(72) As a consequence of Proposition 7 we have:

Corollary 9. *Over a curve every vector bundle with connection admits a framing by parallel sections. Over a one-dimensional base every vector bundle with connection admits local trivializations by parallel frames.*

Here the existence of a parallel frame is over the interval parametrizing the curve. Even if the endpoint of the curve agrees with the starting point, the same may not be true for the initial and ending frames. This is why the second statement is only local.

(73) This corollary fails for base spaces which are not one-dimensional, and this leads to geometric interpretations of the curvature. It will turn out that the corollary encodes the fact that on a one-manifold there is no curvature (as every two-form vanishes identically).

(74) We now prove:

Theorem 10. *A vector bundle $E \to B$ with connection ∇ admits local frames consisting of parallel sections if and only if ∇ is flat, i. e. $F^{\nabla} = 0$.*

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.
2. S. Lang, *Differential and Riemannian Manifolds*, Springer Verlag 1995.
3. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag 1983.