Geometry of manifolds I
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20 October 2003

(1) Topological spaces; open and closed sets; bases for the topology; Hausdorffness; continuity. **Topological manifolds** are locally Euclidean spaces that are Hausdorff and have a countable basis for their topology; charts and atlases.
(2) Metric spaces, $\mathbb{R}^n$.

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(3) **Smooth manifolds** are topological manifolds with an atlas whose transition maps are diffeomorphisms of some class C^k, $k \geq 1$. Examples: Euclidean spaces, spheres, tori, $GL(n, \mathbb{R})$,...
(4) Differentiable functions and maps; diffeomorphisms vs. homeomorphisms.
(5) Unions of atlases; equivalence and maximality.
(6) Dimensions of manifolds and smooth invariance of domain.
(7) Every manifold M is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering of M, there is an atlas $\{(V_k, \varphi_k)\}$ such that the covering by the V_k is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball B_3 of radius 3 for all k and the open sets $W_k = \varphi_k^{-1}(B_1)$ cover M.

Proof. We prove first that there is a sequence G_i, $i = 1, 2, \ldots$ of open sets with compact closures, such that the G_i cover M and satisfy
$$\overline{G_i} \subset G_{i+1}$$
for all i. To this end let U_i, $i = 1, 2, \ldots$ be a countable basis of the topology consisting of open sets with compact closures. (This exists because M is second countable, Hausdorff and locally compact.) Set $G_1 = U_1$. Suppose inductively that we have defined
$$G_k = U_1 \cup \ldots \cup U_{j_k} \ .$$
Then let j_{k+1} be the smallest integer greater than j_k with the property that
$$\overline{G_k} \subset U_1 \cup \ldots \cup U_{j_{k+1}} \ ,$$

and define

$$G_{k+1} = U_1 \cup \ldots \cup U_{j_{k+1}} .$$

This defines the sequence G_k as desired.

Now let $\{U_\alpha \mid \alpha \in A\}$ be an arbitrary open covering of M. For every $x \in M$ we can find a chart (V_x, φ_x) at x with V_x contained in one of the U_α and such that $\varphi_x(V_x) = B_3$. Let $W_x = \varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_i} \setminus G_{i-1}$ by finitely many such W_{x_j} such that at the same time the corresponding V_{x_j} are contained in the open set $G_{i+1} \setminus \overline{G_{i-2}}$. Taking all these V_{x_j} as i ranges over the positive integers we obtain the desired atlas. □

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(8) Every open covering of a smooth manifold admits a subordinate smooth **partition of unity**. This follows from paracompactness and the existence of smooth cutoff functions.
(9) **Vector bundles**; the **tangent bundle** of a smooth manifold, see [1] Section 3.3.

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(10) **Derivatives of smooth maps** between manifolds; tangent vectors as derivatives of curves.
(11) Integration of **vector fields**; **local** and **global flows**; **completeness**, see [1] Section 4.1.

3 November 2003

(12) Every vector field with compact support is complete.
(13) Vector fields as **derivations** and the **Lie derivative**.

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(14) **Commutators** and the Lie algebra structure on the space of vector fields, see [1] Section 4.2.

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(15) The **tensor algebra** and the **exterior algebra** of a real vector space. (See [1] Section 7.2.).

(16) Multilinear algebra of vector bundles and **differential forms**.

13 November 2003

(17) The **exterior derivative**.
(18) **Pullback** of forms, contraction and the **Lie derivative** on forms, Cartan's formula.

17 November 2003

(19) **Closed** and **exact** p-froms; the **de Rham cohomology algebra**.
(20) Existence and uniqueness of the **integral** of n-forms with compact support on oriented n-manifolds. (See [1] Section 8.2.)

20 November 2003

No lecture due to the demonstration against funding cuts in universities.

24 November 2003

(21) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \ .$$

(See [1] Section 8.2.)
(22) The **Poincaré lemma**: every closed form is locally exact.

27 November 2003

(23) The de Rham cohomology of S^n.

1 December 2003

(24) For any closed oriented n-dimensional manifold M without boundary, we have $H^n_{dR}(M) = \mathbb{R}$.
(25) Using the above, we prove Moser's result on **isotopy of volume forms**: if M is closed and oriented, then any two volume forms with the same total volume are diffeomorphic to each other by a diffeomorphism isotopic to the identity.

4 December 2003

(26) We now begin the discussion of **connections** or **covariant derivatives** on vector bundles.

Let $E \to M$ be a smooth vector bundle of rank k over a smooth manifold M of dimension n.

Definition 1. A connection on E is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(E) \longrightarrow \Omega^1(E) \tag{1}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s) \tag{2}$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*M \otimes E)$ is the space of 1-forms on M with values in E. One can evaluate the 1-form on a vector field X to obtain

$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E)\ . \tag{3}$$

The value of $\nabla_X(s)$ at a point $p \in M$ depends on X only through $X(p) \in T_pM$. The dependence on s is more interesting.

Lemma 2. *If s vanishes in some open neighbourhood of p, then so does $\nabla(s)$.*

This shows that ∇ does not increase supports. It is therefore a local differential operator, which can be described in a local trivialisation of E by partial derivatives with respect to local coordinates in M.

Lemma 3. (a) *If ∇_0 and ∇_1 are connections on E, then so is $(1-t)\nabla_0 + t\nabla_1$ for all $t \in \mathbb{R}$.*

(b) *The difference $\nabla_0 - \nabla_1$ is linear with respect to multiplication of sections by arbitrary smooth functions, and is therefore a 1-form with values in $End(E)$, i. e. a section of $T^*M \otimes End(E) = T^*M \otimes E^* \otimes E$.*

Definition 4. A (local) frame for E is a set of smooth sections $s_1, \dots, s_k$ defined over some open set $U \subset M$, whose values are linearly independent at every point $p \in U$.

Thus a set of k local smooth sections $s_1, \dots, s_k$ is a frame if and only if $s_1(p), \dots, s_k(p)$ is a basis of $E_p = \pi^{-1}(p)$ for every $p \in U$. Therefore a frame defined over U defines a trivialization of $E|_U$, and, conversely, every such trivialization

$$\psi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k \tag{4}$$

defines a local frame by setting $s_i(p) = \psi^{-1}(p, e_i)$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$.

Using a partition of unity subordinate to an open cover of M by open sets over which E is trivial, we can prove the existence part of the following:

Proposition 5. *Every vector bundle E admits connections. The space of all connections on E is an affine space for the space $\Omega^1(End(E)) = \Gamma(T^*M \otimes E^* \otimes E)$ of 1-forms with values in $End(E)$.*

The second part of the Proposition follows from the second part of Lemma 3.

(27) We now extend the differential operator given by a connection ∇ to bundle-valued forms of higher degree.

Lemma 6. *For every connection ∇ on $E \to M$ there is a unique $\mathbb{R}$-linear map $\bar{\nabla}\colon \Omega^k(E) \longrightarrow \Omega^{k+1}(E)$ which satisfies*

$$\bar{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla(s) \tag{5}$$

for all $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$. This operator satisfies

$$\bar{\nabla}(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f\bar{\nabla}(\omega \otimes s) \ . \tag{6}$$

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(28) Considering the operator $\bar{\nabla} \circ \nabla\colon \Omega^0(E) \longrightarrow \Omega^2(E)$ associated with a connection ∇ on E. It turns out that this is linear over $C^\infty(M)$, and is therefore given by an element $F^\nabla \in \Omega^2(End(E))$. This is called the **curvature** of ∇.

(29) Fix a local frame $s_1, \ldots, s_k$ for the restriction of E to a trivialising open set in M. With respect to this frame a connection ∇ can be expressed through a matrix $\omega = (\omega_{ij})$ of 1-forms by writing

$$\nabla s_i = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

Then a calculation gives

$$F^\nabla s_i = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} \ .$$

We can write this briefly as $\Omega = d\omega - \omega \wedge \omega$, where the wedge product on the right-hand-side includes matrix multiplication, and is therefore not trivial unless $k = 1$.

Similarly we compute $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$. This is the **Bianchi identity**.

Finally, suppose we have another frame $s'_1, \dots, s'_k$ on the same domain of definition as the original frame. Let ω' and Ω' denote the connection and curvature matrices with respect to this new frame. If

$$s'_i = \sum_{i=1}^{k} g_{ij} s_j \ ,$$

we find the following relations[1]: $\omega' = dg\ g^{-1} + g\omega g^{-1}$ and $\Omega' = g\Omega g^{-1}$, where $g = (g_{ij})$.

11 December 2003

(30) If $E \to M$ is a vector bundle with a connection ∇, we say that a section $s \in \Gamma(E)$ is **parallel** with respect to ∇ if $\nabla s = 0$. In the special case that ∇ is the connection given by some trivialization, a section is parallel if and only if it is constant in the given trivialization. Thus parallel sections should be thought of as the analogs of constant sections for nontrivial bundles.

Proposition 7. *Let $\pi\colon E \to M$ be a vector bundle with a connection ∇, and $c\colon [0,1] \to M$ a smooth curve in the base space. Then for every $v \in \pi^{-1}(c(0))$ there is a unique smooth curve $s\colon [0,1] \to E$ with $\pi \circ s = c$, $s(0) = v$ and $\nabla_{\dot c} s = 0$. Moreover, the map $v \mapsto s(1)$ defines an isomorphism of vector spaces $\pi^{-1}(c(0)) \to \pi^{-1}(c(1))$.*

Here s can be thought of as a section of the pullback bundle $c^*E \to [0,1]$ that is parallel with respect to the given connection. Over M the condition $\nabla_{\dot c} s = 0$ makes sense although s is not a connection over all of M because the covariant derivative is only considered in the direction of c, where s is defined.

The Proposition follows from the existence and uniqueness of the solutions of systems of linear ordinary differential equations with given initial conditions, together with the linear dependence of the solutions on the initial values.

[1] In physics g is called a gauge transformation, and these formulae show how connection and curvature matrices behave under gauge transformations. The curvature is more invariant than the connection...

Corollary 8. *Over a curve every vector bundle with connection admits a framing by parallel sections.*

(31) The last corollary fails for base spaces which are not one-dimensional, and this leads to geometric interpretations of the curvature. It will turn out that the corollary encodes the fact that on a one-manifold there is no curvature (as every two-form vanishes identically).

Recall that with respect to a framing $s_1, \dots, s_k$ of E a connection ∇ is expressed by the matrix (ω_{ij}) of one-forms. If we choose a chart in M with local coordinates $y_1, \dots, y_n$, then in the domain of this chart every one-form can be expressed uniquely as a linear combination of the dy_i. In particular, there are smooth functions ω_{ij}^α on the domain of the chart such that

$$\omega_{ij} = \sum_{\alpha=1}^{n} \omega_{ij}^\alpha dy_\alpha \ . \tag{7}$$

Denoting the vector fields $\frac{\partial}{\partial y_\alpha}$ by ∂_α, we find the following:

$$\nabla_{\partial_\alpha} s_i = \langle \partial_\alpha, \nabla s_i \rangle = \sum_{j=1}^{k} \langle \partial_\alpha, \omega_{ij} \rangle s_j = \sum_{j=1}^{k} \omega_{ij}^\alpha s_j \ .$$

More generally, if

$$s = \sum_{i=1}^{k} x_i s_i \ ,$$

then

$$\nabla_{\partial_\alpha} s = \sum_{j=1}^{k} \Big(\frac{\partial x_j}{\partial y_\alpha} + \sum_{i=1}^{k} x_i \omega_{ij}^\alpha \Big) s_j \ .$$

Writing A^α for the matrix (ω_{ij}^α) of functions we see that the operator ∇_{∂_α}, which we abbreviate to ∇_α, has the form $\nabla_\alpha = \partial_\alpha + A^\alpha$.

We can now characterize flatness of ∇ through the condition that the directional derivatives ∇_α commute:

Proposition 9. *The connection ∇ is flat if and only if $[\nabla_\alpha, \nabla_\beta] = 0$ for every local coordinate system $y_1, \dots, y_n$ on the base manifold M.*

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(32) We now prove:

Theorem 10. *A vector bundle $E \to M$ with connection ∇ admits local frames consisting of parallel sections if and only if ∇ is flat, i. e. $F^\nabla = 0$.*

(33) As a consequence of the previous theorem we obtain:

Corollary 11. *A vector bundle $E \to M$ admits a flat connection if and only if it admits a system of local trivializations*

$$\psi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

for which all transition maps

$$\begin{aligned} \psi_j \circ \psi_i^{-1} \colon (U_i \cap U_j) \times \mathbb{R}^k &\longrightarrow (U_i \cap U_j) \times \mathbb{R}^k \\ (x, v) &\longmapsto (x, g_{ij}(x)v) \end{aligned}$$

have the property that the functions $g_{ij} \colon U_i \cap U_j \to GL_k(\mathbb{R})$ are constant.

18 December 2003

(34) **Metrics** on vector bundles are smoothly varying fiberwise scalar products. Using a partition of unity we prove that every vector bundle admits a metric.

(35) A connection ∇ on a vector bundle $E \to M$ is compatible with a metric $\langle\ ,\ \rangle$ if and only if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

for all pairs of sections $s_1, s_2 \in \Gamma(E)$. Sometimes a connection compatible with some metric is called a metric connection.

Lemma 12. *A connection ∇ is compatible with a metric $\langle\ ,\ \rangle$ if and only if its connection matrix ω with respect to any local frame that is orthonormal with respect to $\langle\ ,\ \rangle$ is skew-symmetric, i. e. $\omega_{ij} = -\omega_{ji}$ for all i and j. In this case the curvature matrix Ω with respect to a local orthonormal frame is also skew-symmetric: $\Omega_{ij} = -\Omega_{ji}$.*

Finally, metric connections always exist. The following is proved combining the above lemma with the proof of Proposition 5.

Proposition 13. *Every vector bundle E with a metric $\langle\ ,\ \rangle$ admits compatible connections. The space of all compatible connections is an affine space for the space $\Omega^1(SkewEnd(E))$ of 1-forms with values in the endomorphisms of E which are skew-symmetric with respect to $\langle\ ,\ \rangle$.*

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(36) We now consider vector bundles of small rank equipped with metric connections. If the rank is $= 1$, then the skew-symmetry of the connection matrix shows that every metric connection is flat. As every bundle admits a metric and a compatible connection, we conclude that all rank one bundles admit flat connections. In some sense, rank $= 2$ is the first interesting case.

(37) Suppose $E \to M$ is a smooth vector bundle of rank $= 2$ equipped with a metric. In addition, we assume that E is oriented, i. e. we have chosen a system of local trivialisations for which all transition maps take values in the orientation-preserving linear maps only. We work only with oriented orthonormal local frames s_1 and s_2. Due to the orientation, equivalently the ordering of the orthonormal frames, we know unambiguously what the curvature form Ω_{12} is[2]. Passing to a different orthonormal frame by an orientation-preserving gauge transformation changes the curvature matrix Ω by conjugating it with an element of $SO(2)$, which turns out not to be a change at all. So Ω_{12} is in fact a globally defined 2-form, which is independent of the choice of frame. Moreover, it is locally exact $\Omega_{12} = d\omega_{12}$, and therefore closed. Thus the following definition makes sense:

Definition 14. The **Euler class** of the oriented bundle $E \to M$ is the de Rham cohomology class $e(E, \nabla) = \frac{-1}{2\pi}[\Omega_{12}] \in H^2_{dR}(M)$.

Here ∇ is any connection compatible with the given metric $\langle\ ,\ \rangle$. We have the following list of properties:

- If ∇ is flat, then $e(E, \nabla) = 0$.
- Denoting by $\bar{E}$ the vector bundle obtained from the oriented bundle E by changing the orientation, we have $e(\bar{E}, \nabla) = -e(E, \nabla)$.
- If we change the connection ∇ to a different connection ∇' compatible with the same metric, then ∇' has $\omega_{12} + a$ in its connection matrix where ∇ had ω_{12}, with a a globally defined 1-form on M. The Euler class with respect to this new connection is $e(E, \nabla') = [\Omega_{12} + da] = [\Omega_{12}] = e(E, \nabla)$.

Thus, we see that the Euler class, as a cohomology class, is independent of the connection we choose, as long as it is compatible with the metric. Therefore, we write $e(E, \langle\ ,\ \rangle)$ for the Euler

[2] Changing the order of the frame changes the sign due to the skew-symmetry.

class. We will see later, that it is in fact independent of the metric, and is therefore a topological invariant of the oriented bundle E.

(38) We consider $M = S^2$, covered by two charts diffeomorphic to $\mathbb{R}^2$. Construct an orientable vector bundle E of rank 2 over S^2 by making an identification of $(x, v) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 = X_1$ with $(-\frac{x}{|x|^2}, g(x) \cdot v) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 = X_2$ for some smooth map $g \colon \mathbb{R}^2 \setminus \{0\} \to SO(2)$. On these two trivializations we have metrics defined by the standard scalar product on $\mathbb{R}^2$, and as g is assumed to take values in the orthogonal group, we obtain an induced metric on E. We take the two flat connections induced by the two trivializations and glue them together with an explicit partition of unity to obtain a metric connection on E.

In this case an orientation of S^2 gives us an identification of $H^2_{dR}(S^2)$ with $\mathbb{R}$, so that the Euler class is simply the number obtained by integration of $\frac{-1}{2\pi}\Omega_{12}$ over S^2 (with respect to the chosen orientation of S^2). Identifying $\mathbb{R}^2 \setminus \{0\}$ with $S^1 \times \mathbb{R}$ with coordinates (θ, t), we specify $g(\theta, t)$ to be rotation by angle $n\theta$ (independently of t, for some integer n). Then we obtain Euler class $= \pm n$ for our bundle, depending on how we orient E and S^2. (When $n = 0$, the bundle is trivial and the connection is flat.)

8 January 2004

(39) An **affine connection** on a smooth manifold M is a connection ∇ on its tangent bundle $TM \to M$. For affine connections the variables X and s in $\nabla_X s$ are on equal footing, as they are both sections of the tangent bundle. This leads to possible symmetries which make no sense in the more general setting of arbitrary vector bundles.

(40) The **torsion** of an affine connection ∇ is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \mathcal{X}(M)$.

Lemma 15. *The torsion defines a skew-symmetric map*

$$T \colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

that is bilinear over $C^\infty(M)$.

An affine connection ∇ is called **symmetric** if it is torsion-free, i. e. T vanishes identically[3].

(41) To explain why torsion-freeness is indeed a symmetry condition, we consider the expression of an affine connection in a local coordinate system $(y_1, \ldots, y_n)$ on M. We write ∂_i for the coordinate vector fields $\frac{\partial}{\partial y_i}$, and use the local frame $\partial_1, \ldots, \partial_n$. Then

$$\nabla \partial_i = \sum_{j=1}^{n} \omega_{ij} \otimes \partial_j \ ,$$

and using (7) we obtain

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \omega_{jk}^{i} \partial_k \ ,$$

which is usually written as

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} \partial_k$$

in classical notation. Therefore, we define the **Christoffel symbols** of the affine connection ∇ with respect to the coordinate system $(y_1, \ldots, y_n)$ to be $\Gamma_{ij}^k = \omega_{jk}^i$.

Returning to the definition of torsion, we see that

$$T(\partial_i, \partial_j) = \sum_{k=1}^{n} (\omega_{jk}^{i} - \omega_{ik}^{j}) \partial_k = \sum_{k=1}^{n} (\Gamma_{ij}^{k} - \Gamma_{ji}^{k}) \partial_k \ .$$

As the torsion is linear over the smooth functions, we obtain the following:

Lemma 16. *An affine connection ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any local coordinate system.*

Thus symmetry of the connection really refers to a symmetry of the Christoffel symbols expressing this connection in local coordinates.

(42) Consider a **Riemannian manifold** M, that is a smooth manifold with a metric on its tangent bundle. In this case it turns out that there is a unique symmetric connection that is at the same time compatible with the given metric:

Proposition 17 (Fundamental Lemma of Riemannian Geometry)**.** *The tangent bundle of a Riemannian manifold admits a unique torsion-free connection compatible with the metric.*

[3] Note that requiring the naive symmetry $\nabla_X Y = \nabla_Y X$ for all X and Y leads to a contradiction.

Proof. Given vector fields X, Y and Z, we can use the two requirements, compatibility with the metric $\langle\ ,\ \rangle$ and torsion-freeness, to conclude that the only possible value for $\langle \nabla_X Y, Z\rangle$ is

$$\begin{aligned}(8)\quad \langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y],Z\rangle + \langle [Z,X],Y\rangle + \langle [Z,Y],X\rangle +\\ L_X\langle Y,Z\rangle + L_Y\langle X,Z\rangle - L_Z\langle X,Y\rangle)\ .\end{aligned}$$

This proves uniqueness. To see existence, we use (8) as a definition. As $\langle\ ,\ \rangle$ is non-degenerate, requiring that the equation hold for all Z uniquely defines $\nabla_X Y$. We then check that this ∇ is indeed a connection, is metric-compatible, and torsion-free. □

12 January 2004

(43) If $E \to M$ is a vector bundle with a connection ∇, then the dual bundle $E^* \to M$ carries a well-defined dual connection ∇^* characterized by the identity

$$d\langle s, \alpha\rangle = \langle \nabla s, \alpha\rangle + \langle s, \nabla^* \alpha\rangle$$

for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$. (The brackets here denote the natural pairing between a bundle and its dual bundle, not a metric.)

In the case of an affine connection ∇, the dual connection ∇^* gives us the following characterization of torsion-freeness:

Proposition 18. *An affine connection ∇ on M is torsion-free if and only if the exterior differential on one-forms is given by the composition*

$$\Gamma(T^*M) \xrightarrow{\nabla^*} \Gamma(T^*M \otimes T^*M) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*M)\ .$$

(44) Next we discuss a concrete example where we can write down the Levi-Civita connection of a Riemannian metric explicitly. We shall use this to compute the Euler class of the tangent bundle of the 2-sphere.

Consider the unit sphere $S^2 \subset \mathbb{R}^3$. We take the standard scalar product on $\mathbb{R}^3$. By restriction to the tangent spaces to the sphere we obtain a Riemannian metric on S^2. Next choosing an explicit parametrization of an open set on the sphere by an open set in $\mathbb{R}^2$ we construct an explicit local orthonormal frame s_1, s_2 for the tangent bundle. Let ∇_0 be the Levi-Civita connection of $\mathbb{R}^3$ considered as a Riemannian manifold. This is the flat connection for which the standard global frame of $\mathbb{R}^3$

is parallel. We define an affine connection ∇ on S^2 by taking covariant derivatives in $\mathbb{R}^3$ using ∇_0 and then projecting to TS^2 along the orthogonal complement of $TS^2 \subset T\mathbb{R}^3$. Computing the connection matrix with respect to our chosen orthonormal frame, we see that it is skew-symmetric, and so ∇ is compatible with the metric of S^2. (In fact, ∇ is also torsion-free, and therefore is the Levi-Civita connection, but we will not need this here.)

We define the orientation of S^2 so that s_1, s_2 is positively oriented. Our calculation of the connection matrix of ∇ with respect to s_1, s_2 yields $\Omega_{12} = s_2^* \wedge s_1^*$. Therefore, the Euler class $e(TS^2, \langle\ ,\ \rangle)$, identified with a number by using the isomorphism $H^2_{dR}(S^2) = \mathbb{R}$ given by the orientation, turns out to be:

$$e(TS^2, \langle\ ,\ \rangle) = \frac{-1}{2\pi}\int_{S^2} s_2^* \wedge s_1^* = \frac{1}{2\pi}\int_{S^2} dvol = \frac{1}{2\pi}4\pi = 2\ .$$

Note that this is independent of the orientation we have chosen, because if we change the orientation, then both the Euler class and the isomorphism $H^2_{dR}(S^2) = \mathbb{R}$ change by a sign, and the two signs cancel.

15 January 2004

(45) Let M be an oriented smooth manifold of dimension 2, i. e. a surface. For a Riemannian metric $\langle\ ,\ \rangle$ on M we consider the unique function $\mathcal{K}$ defined by the equation

$$-\Omega_{12} = \mathcal{K} dvol\ ,$$

where Ω_{12} is the curvature 2-form of the Levi-Civita connection with respect to a local oriented orthonormal frame s_1, s_2 and $dvol$ is the Riemannian volume form $s_1^* \wedge s_2^*$ defined by the metric and the orientation. The function $\mathcal{K}$ is smooth, and is called the **Gaussian curvature** of $\langle\ ,\ \rangle$ on M. The following Theorem asserts that on a closed surface the integral of the Gaussian curvature is independent of the metric (and the orientation) that we have chosen:

Theorem 19 (Gauss–Bonnet Theorem). *Let M be a compact oriented surface without boundary. For any Riemannian metric we have*

$$\frac{1}{2\pi}\int_M \mathcal{K} dvol = e(M)\ ,$$

where $e(M)$*, the Euler number of* M*, is defined by mapping the Euler class* $e(TM) \in H^2_{dR}(M)$ *to* $\mathbb{R}$ *via the isomorphism given by the orientation of* M*.*

Proof. Our definitions are such that the theorem is tautological. We defined the Euler class of TM to be the cohomology class represented by the closed form $\frac{-1}{2\pi}\Omega_{12}$, and we proved that this is independent of the metric. Therefore, the image of $e(TM) \in H^2_{dR}(M)$ in $\mathbb{R}$ depends only on the orientation of M; but if we change the orientation, then the Euler class and the isomorphism $H^2_{dR}(M) = \mathbb{R}$ both change by a sign, and the two signs cancel. $\square$

Recall that we calculated a specific example for $M = S^2$ with the metric induced from $\mathbb{R}^3$. In this case the Gaussian curvature is constant and $= 1$. The Gauss–Bonnet theorem shows that for any other metric on S^2 the average of the Gaussian curvature is also positive. For another example consider $M = T^2$. In this case we can choose a global trivialization of the tangent bundle given by two non-vanishing vector fields which are everywhere linearly independent, for example the vector fields tangent to the factors in a decomposition $T^2 = S^1 \times S^1$. We can define a metric by declaring the two chosen vector fields to be orthonormal. Then, by definition, this metric is flat, and the Euler number of the torus vanishes. Therefore, the average of the Gaussian curvature of any metric on T^2 vanishes, even for non-flat metrics.

(46) The Gauss–Bonnet theorem was tautological, but we can endow it with non-trivial content by identifying the Euler number with other quantities, defined in a different way. We shall do this by generalizing the above discussion for the torus.

Let X be a vector field with isolated zeros on a compact oriented surface M without boundary. Let $p_1, \ldots, p_k$ be the zeros, and $p_i \in B_i \subset M$ disjoint neighbourhoods diffeomorphic to open balls. Set

$$N = M \setminus \left(\bigcup_{i=1}^{k} B_i\right) .$$

Choose a metric $\langle\ ,\ \rangle$ on TM whose restriction to the B_i is flat. On N we define a frame by setting $X_1 = X/|X|$, and then choosing for X_2 the unique vector field which completes X_1 to a positively oriented orthonormal frame. Let ω_{12} and Ω_{12} be connection and curvature forms with respect to this frame. As Ω_{12} vanishes on the B_i we have

$$e(M) = \frac{1}{2\pi}\int_M \mathcal{K} dvol = \frac{-1}{2\pi}\int_M \Omega_{12} = \frac{-1}{2\pi}\int_N \Omega_{12} \ .$$

By Stokes's Theorem this integral equals

$$\frac{-1}{2\pi}\int_{\partial N} \omega_{12} = \sum_{i=1}^{k} \frac{1}{2\pi}\int_{\partial B_i} \omega_{12} \ ,$$

with the change in sign due to the change in orientation depending on whether a given circle is viewed as being in the boundary of N or of B_i.

Consider the vector field X_1 as a map $X_1 \colon \partial B_i \to S^1$, where the target is the unit circle in $\mathbb{R}^2$ considered as the tangent space to the two-dimensional disk. Then we calculate that the pullback of the standard volume form on S^1 is precisely the connection form with respect to X_1, X_2:

$$X_1^* d\theta = \omega_{12} \ .$$

Thus,

$$\frac{1}{2\pi}\int_{\partial B_i} \omega_{12} = \frac{1}{2\pi}\int_{\partial B_i} X_1^* d\theta = \frac{1}{2\pi} deg(X_1|_{\partial B_i}) \int_{S^1} d\theta = deg(X_1|_{\partial B_i}) \ .$$

Thus we have proved the following:

Theorem 20 (Poincaré–Hopf Theorem)**.** *Let M be a compact oriented surface without boundary. If X is a vector field with isolated zeros $p_1, \dots, p_k$ on M, then there are integers $Ind(X, p_i)$ depending on the vector field X such that*

$$\frac{1}{2\pi}\int_M \mathcal{K} dvol = e(M) = \sum_{i=1}^{k} Ind(X, p_i) \ .$$

In particular, the index sum on the right-hand-side is independent of the vecctor field X.

The indices $Ind(X, p_i)$ of the zeroes are defined to be the winding numbers given by the degrees $deg(X_1|_{\partial B_i})$.

19 January 2004

(47) We discuss examples of indices of zeros of vector fields in the plane and on compact manifolds. As a corollary of the Poincaré–Hopf Theorem 20 we find that the Euler number of a closed

oriented surface if genus g is $2-2g$. This checks with the calculations we have done for the sphere and the torus, where $g=0$ and $g=1$ respectively.

(48) We calculate the Gaussian curvature explicitly for the metric induced on T^2 by an embedding in $\mathbb{R}^3$ as a surface of revolution.

22 January 2004

(49) Our calculations for $S^2 \subset \mathbb{R}^3$ and for $T^2 \subset \mathbb{R}^3$ in (44) and in (48) above involve special cases of the following:

Lemma 21. *Let $(N, \langle\ ,\ \rangle)$ be a Riemannian manifold, and $M \subset N$ a submanifold. The Levi-Civita connection ∇^M of the metric on M obtained by restricting $\langle\ ,\ \rangle$ to the subbundle $TM \subset TN|_M$ is given by*

$$\nabla^M_X Y = \pi \nabla^N_{\tilde{X}} \tilde{Y} \ ,$$

where $\tilde{X}$ and $\tilde{Y}$ are local extensions of $X, Y \in \mathcal{X}(M)$ to N, ∇^N is the Levi-Civita connection of $(N, \langle\ ,\ \rangle)$, and $\pi\colon TN|_M \to TM$ is the projection along the orthogonal complement of $TM \subset TN|_M$ with respect to $\langle\ ,\ \rangle$.

(50) Suppose now that M has codimension one in N. If M and N are oriented, then the normal bundle inherits an induced orientation. We choose a positively oriented local orthonormal frame $s_1, \ldots, s_n$ for TN at $p \in M$ in such a way that $s_1, \ldots, s_{n-1}$ is a positively oriented orthonormal frame for TM, and s_n is a positively oriented frame for the normal space.

Definition 22. The **Weingarten map** at $p \in M$ is the linear map

$$\begin{aligned} L\colon T_pM &\longrightarrow T_pM \\ v &\longmapsto \nabla^N_v s_n \end{aligned}$$

This has the following easily proved property, see [1], Section 10.3.

Lemma 23. *The Weingarten map is symmetric with respect to the metric, i. e. $\langle L(X), Y\rangle = \langle X, L(Y)\rangle$ for all $X, Y \in T_pM$.*

(51) For $X, Y, Z \in T_pM$ we can relate the values $F^{\nabla^N}(X,Y)Z$ and of $F^{\nabla^M}(X,Y)Z$ of the curvatures of ∇^N and of ∇^M through the Weingarten map. In the special case that ∇^N is flat we obtain the following classical identities:

$$F^{\nabla^M}(X,Y)Z = \langle L(Y), Z\rangle\ L(X) - \langle L(X), Z\rangle\ L(Y) \ ,$$

$$L([X,Y]) = \nabla^M_X L(Y) - \nabla^M_Y L(X) \ .$$

The first of these is the so-called **Gauss equation**, and the second is the **Codazzi–Mainardi equation**[4].

26 January 2004

(52) Let $M \subset \mathbb{R}^n$ be an oriented smooth hypersurface. We orient $\mathbb{R}^n$ through the standard basis. The **Gauss map**

$$\begin{aligned} G \colon M &\longrightarrow S^{n-1} \\ p &\longmapsto G(p) \end{aligned}$$

is defined by assigning to each point of M the unit normal vector $G(p)$ to M for which $s_1(p), \ldots, s_{n-1}(p), G(p)$ is a positively oriented frame of $\mathbb{R}^n$ for any positively oriented frame $s_1(p), \ldots, s_{n-1}(p)$ of T_pM.

Lemma 24. *There is a natural identification of T_pM with $T_{G(p)}S^{n-1}$. Under this identification, the derivative of the Gauss map at p coincides with the Weingarten map.*

Proof. Both T_pM and $T_{G(p)}S^{n-1}$ can be identified with the orthogonal complement of $G(p)$ in $\mathbb{R}^n$. The derivative of the Gauss map can be calculated from the definition, see [1], Section 10.3. □

(53) Assume now that $n = 3$. In this case we can compute the curvature of the surface M from the Gauss map in a particularly straightforward manner:

Theorem 25. *The curvature form Ω^M_{12} of the Levi-Civita connection of the metric induced on M from the scalar product on $\mathbb{R}^3$ is given by $-G^* dvol_{S^2}$, where $dvol_{S^2}$ is the Riemannian volume form of the unit sphere.*

Proof. Choose an oriented local orthonormal frame s_1, s_2 for TM, and complete it with $s_3(p) = G(p)$ to an oriented orthonormal frame for $\mathbb{R}^3$. Let ω_{ij} be the connection matrix of the Levi-Civita connection of $\mathbb{R}^3$ with respect to this frame. Then, according to Lemma 21 the upper left-hand two-by-two submatrix of ω_{ij} is the connection matrix of the Levi-Civita connection of the induced metric on M.

From the definition of the Gauss map we calculate

$$G^* dvol_{S^2}(s_1, s_2) = \omega_{13} \wedge \omega_{23}(s_1, s_2) \ .$$

[4]See [1], Section 10.3, but beware of the sign error in the Codazzi–Mainardi equation.

Thus $G^*dvol_{S^2} = \omega_{13} \wedge \omega_{23}$. On the other hand, the flatness of $\mathbb{R}^3$ gives $d\omega_{12} - \omega_{13} \wedge \omega_{32} = 0$, as $\Omega_{12}^{\mathbb{R}^3} = 0$. Combining the two, we obtain:

$$\Omega_{12}^M = d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23} = -G^*dvol_{S^2} \ .$$

□

The Gauss curvature $\mathcal{K}$ was defined by the equation $-\Omega_{12}^M = \mathcal{K} dvol_M$. Thus, we obtain

$$\mathcal{K} = \frac{G^*dvol_{S^2}}{dvol_M} \ ,$$

showing that the Gauss curvature measures the distortion of signed area under the Gauss map. In particular,

Corollary 26. *For $p \in M \subset \mathbb{R}^3$ the Gauss curvature $\mathcal{K}(p)$ is non-zero if and only if the Gauss map is a local diffeomorphism at p. In this case the sign of $\mathcal{K}(p)$ is positive or negative depending on whether G is orientation-preserving or reversing at p.*

The sign of the curvature can be seen by checking whether, locally near p, M lies entirely on one side of the tangent plane T_pM as an affine subspace of $\mathbb{R}^3$ through the point p.

Integrating the equality in Theorem 25 we obtain:

Corollary 27. *Let $M \subset \mathbb{R}^3$ be a closed oriented surface. Its Euler number $e(M)$ is twice the degree of the Gauss map.*

Proof. By definition and by Theorem 25 we have:

$$\begin{aligned} e(M) &= \frac{-1}{2\pi} \int_M \Omega_{12}^M = \frac{1}{2\pi} \int_M G^*dvol_{S^2} \\ &= \frac{1}{2\pi} deg(G) \int_{S^2} dvol_{S^2} = \frac{1}{2\pi} deg(G)\ 4\pi = 2deg(G) \ . \end{aligned}$$

□

(54) Here is yet another way to calculate the Gauss curvature of a surface:

Lemma 28. *Let s_1, s_2 be a positively oriented orthonormal frame on an oriented surface M endowed with a Riemannian metric $\langle \ , \ \rangle$ with Levi-Civita connection ∇. Then*

$$\mathcal{K} = \langle F^\nabla(s_1, s_2)s_2, s_1 \rangle \ .$$

Proof. For a surface in $\mathbb{R}^3$ we can argue as follows. We know from Theorem 25 and Lemma 24 that $\mathcal{K}$ is the determinant of the Weingarten map, and this is easily seen to equal $\langle F^\nabla(s_1, s_2)s_2, s_1\rangle$.

In general, for an abstract surface, we calculate using

$$F^\nabla(s_1, s_2)s_2 = \nabla_{s_1}\nabla_{s_2}s_2 - \nabla_{s_2}\nabla_{s_1}s_2 - \nabla_{[s_1,s_2]}s_2 \ .$$

This leads to $\langle F^\nabla(s_1, s_2)s_2, s_1\rangle = (-d\omega_{12})(s_1, s_2)$. As $-d\omega_{12} = \mathcal{K}s_1^* \wedge s_2^*$, the claim follows. □

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.