Geometry of manifolds II
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(1) In this course we shall assume a basic knowledge of smooth manifolds and smooth vector bundles.

(2) First we recall the notion of a connection on a vector bundle. Let $E \to B$ be a differentiable vector bundle of rank k over a smooth manifold B of dimension n.

Definition 1. A **connection** on E is an $\mathbb{R}$-linear map

$$\nabla \colon \Gamma(E) \longrightarrow \Omega^1(E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for all $f \in C^\infty(B)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*B \otimes E)$ is the space of 1-forms on B with values in E. One can evaluate the 1-form on a vector field X to obtain

$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E) \ .$$

The Leibniz rule is then equivalent to

$$\nabla_X(fs) = (L_X f)s + f\nabla_X(s)$$

for all $X \in \mathcal{X}(B)$.

(3) If $E \to B$ is a vector bundle with a connection ∇, then the dual bundle $E^* \to B$ carries a well-defined **dual connection** ∇^* characterized by the identity

$$d\langle s, \alpha\rangle = \langle \nabla s, \alpha\rangle + \langle s, \nabla^*\alpha\rangle$$

for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$. (The brackets here denote the natural pairing between a bundle and its dual bundle, not a metric.)

(4) In the definition of a connection, see (2) above, X is a section of $TB \longrightarrow B$ and s is a section of $E \longrightarrow B$. Consider now the case when the two bundles coincide, so E is just the tangent bundle of a smooth manifold M. An **affine connection** on M is a connection on its tangent bundle. For affine connections the variables X and s in $\nabla_X s$ are on equal footing, as they are both sections of the tangent bundle. This leads to possible symmetries which make no sense in the more general setting of arbitrary vector bundles.

(5) The **torsion** of an affine connection ∇ is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \mathcal{X}(M)$.

Lemma 2. *The torsion defines a skew-symmetric map*

$$T^\nabla \colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

that is bilinear over $C^\infty(M)$.

This means that we can think of the torsion T^∇ as a two-form on M with values in the tangent bundle: $T^\nabla \in \Omega^2(TM)$.

(6) If we choose a reference connection ∇_0 on TM, then the affine space of connections on $TM \longrightarrow M$ is identified with its vector space of translations $\Omega^1(End(TM))$. The torsion then becomes a map

$$\begin{aligned} T\colon \Omega^1(End(TM)) &\longrightarrow \Omega^2(TM) \\ A &\longmapsto T^{\nabla_0+A} \,. \end{aligned}$$

Note that

$$T^{\nabla_0+A}(X,Y) = T^{\nabla_0}(X,Y) + (A(X))Y - (A(Y))X,$$

so that when adding A, the torsion changes by a certain skew-symmetrization of A.

(7) An affine connection ∇ is called **symmetric** if it is torsion-free, i. e. T^∇ vanishes identically[1].

To explain why torsion-freeness is a symmetry condition, we consider the expression of an affine connection in a local coordinate system $(x_1,\ldots,x_n)$ on M. We write ∂_i for the coordinate vector fields $\frac{\partial}{\partial x_i}$, and use the local frame $\partial_1,\ldots,\partial_n$. Then

$$\nabla \partial_i = \sum_{j=1}^n \omega_{ij} \otimes \partial_j \ ,$$

with the connection matrix

$$\omega_{ij} = \sum_{k=1}^n \omega_{ij}^k dx_k \ .$$

This gives

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \omega_{jk}^i \partial_k \ ,$$

which is usually written as

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k$$

in classical notation. Therefore, we define the **Christoffel symbols** of the affine connection ∇ with respect to the coordinate system $(x_1,\ldots,x_n)$ to be $\Gamma_{ij}^k = \omega_{jk}^i$.

Returning to the definition of torsion, we see that

$$T^\nabla(\partial_i,\partial_j) = \sum_{k=1}^n (\omega_{jk}^i - \omega_{ik}^j)\partial_k = \sum_{k=1}^n (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k \ .$$

As the torsion is linear over the smooth functions, we obtain the following:

Lemma 3. *An affine connection ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any local coordinate system.*

Thus symmetry of the affine connection really refers to a symmetry of the Christoffel symbols expressing this connection in local coordinates.

(8) The dual connection ∇^* gives us the following characterization of torsion-freeness:

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[1]Note that requiring the naive symmetry $\nabla_X Y = \nabla_Y X$ for all X and Y leads to a contradiction.

Proposition 4. *An affine connection ∇ on M is torsion-free if and only if the exterior derivative on one-forms is given by the composition*

$$\Gamma(T^*M) \xrightarrow{\nabla^*} \Gamma(T^*M \otimes T^*M) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*M) \; .$$

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(9) We now begin with the consideration of metrics on manifolds.

Definition 5. A **pseudo-Riemannian metric** on a manifold M is a non-degenerate symmetric bilinear form g defined on all tangent spaces T_pM depending smoothly on p.

A pseudo-Riemannian metric is called **Riemannian** if it is positive definite.

As usual, smooth dependence on p is defined via charts for M. Equivalently it means that for all smooth vector fields X and Y, the function $g(X,Y)$ is also smooth.

(10) Given a pseudo-Riemannian metric g on M, we say that an affine connection ∇ is **compatible** with g if

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y)$$

for all vector fields X and Y. Such a connection will be called metric (with respect to g).

We know that for Riemannian metrics there always exist compatible connections (see the proof for arbitrary positive definite scalar products on arbitrary vector bundles last semester). It is not clear a priori whether this remains true in the pseudo-Riemannian case.

(11) Given a pseudo-Riemannian metric g on M, we can define what it means for an endomorphism of TM to be symmetric or skew-symmetric with respect to g. We write $Skew - End(TM)$ for the vector bundle of skew-symmetric endomorphisms of TM, supressing g in the notation. The difference of two connections that are both compatible with g is a one-form with values in $Skew - End(TM)$. Thus, if there are any connections compatible with g, then they form an affine space whose vector space of translations is $\Omega^1(Skew - End(TM))$.

(12) We now prove the following:

Proposition 6. *On any pseudo-Riemannian manifold (M,g) there is a unique affine connection ∇ that is compatible with g and has a given two-form T with values in TM as its torsion.*

First proof. Let us suppose that there are affine connections compatible with g, and pick one, called ∇_0, as a reference. Then the space of all affine connections compatible with g is identified with

$$\Omega^1(Skew - End(TM)) = \Gamma(T^*M \otimes Skew - End(TM)) \subset \Gamma(T^*M \otimes T^*M \otimes TM) \; .$$

The torsion is a map from this space to

$$\Omega^2(TM) = \Gamma(\Lambda^2 T^*M \otimes TM)$$

sending A to T^{∇_0} plus the skew-symmetrization of A, see (6) above.

Assume that $\nabla_0 + A_1$ and $\nabla_0 + A_2$ have the same torsion. Then we can calculate that $A_1 = A_2$. Thus the torsion map is injective. But, at every point, it is an affine map between the fibers of two vector bundles of the same rank. Thus, if it is injective, it is also surjective. $\square$

(13) We want to give an alternative proof for Proposition 6, which produces a formula for the desired connection and therefore does not assume that connections compatible with g exist. Instead of assuming this, we prove it as a consequence of the following argument.

Second proof. We first prove uniqueness. So let ∇ be an affine connection compatible with g having torsion tensor T. Then using compatibility with g and the formula for the torsion alternately three times, we find that ∇ must be determined by the following formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}(g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) - g(X, T(Y,Z)) + \\ + g(Y, T(Z,X)) + g(Z, T(X,Y)) + L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y))) .$$

To prove uniqueness, we use this equation as a definition for $\nabla_X Y$. This defines a connection, and one can check that it is both compatible with g and has $T^\nabla = T$. $\square$

(14) As a special case of Proposition 6 we note the following:

Corollary 7. *On every pseudo-Riemannian manifold (M,g) there is a unique torsion-free metric affine connection.*

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(15) By Corollary 7 every pseudo-Riemannian manifold (M,g) has a unique torsion-free affine connection compatible with the metric. This connection ∇ is called the **Levi-Cività connection** of (M,g), or of g.

(16) The **Riemann curvature tensor** R of a pseudo-Riemannian manifold (M,g) is defined using its Levi-Cività connection:

$$R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$
$$(X,Y,Z) \longmapsto R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z .$$

Recall that for an arbitrary connection ∇ on a vector bundle E, the curvature F^∇ is a two-form on the base with values in $End(E)$. Here $R = F^\nabla$ for the Levi-Cività connection ∇. The notation $R(X,Y)Z$ means that the two-form is evaluated on X and Y, and the resulting endomorphism is then applied to Z.

(17) The most important properties of the Riemann curvature are the following:
- (i) it is trilinear over $C^\infty(M)$,
- (ii) it is skew-symmetric in the first two arguments,
- (iii) it satisfies the **first Bianchi identity**: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ for all X, Y and Z,
- (iv) $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$ for all X, Y, Z and W, and
- (v) $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$ for all X, Y, Z and W.

(18) Let V be a real vector space with a non-degenerate symmetric bilinear form g (e.g. a tangent space of a pseudo-Riemannian manifold). A subspace $U \subset V$ is called **non-degenerate** if the restriction $g|_U$ is non-degenerate. A non-zero vector $X \in V$ is called **null** if it spans a degenerate subspace.

(19) For a two-dimensional subspace $\sigma \subset V$ non-degeneracy is equivalent to the non-vanishing of $Q(X,Y) = g(X,X)g(Y,Y) - (g(X,Y))^2$, for any basis X and Y of σ.

(20) Now let (M,g) be a pseudo-Riemannian manifold with Riemann tensor R.

Definition 8. For a non-degenerate tangent two-plane σ we define the **sectional curvature** to be

$$K(\sigma) = \frac{g(R(X,Y)Y,X)}{Q(X,Y)} \ .$$

This depends only on g and σ, and not on the basis chosen for σ.

(21) By definition, the sectional curvature of (M,g) is determined by its Riemann tensor. However, the converse is also true:

Proposition 9. *The collection of sectional curvatures for all non-degenerate two-planes $\sigma \subset TM$ determines the Riemann curvature tensor R of (M,g).*

In the proof we use the following:

Lemma 10. *Every pair of vectors X and Y can be approximated arbitrarily closely by X' and Y' such that the span of X' and Y' is a non-degenerate two-plane.*

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(22) The proof of Proposition 9 can easily be adapted to prove the following characterization of spaces with constant sectional curvature:

Proposition 11. *A pseudo-Riemannian manifold (M,g) has sectional curvature equal to a fixed real number $K_0 \in \mathbb{R}$ for all non-degenerate two-planes $\sigma \subset TM$ if and only if the following identity holds for all X, Y, Z and $T \in \mathcal{X}(M)$:*

$$g(R(X,Y)Z,T) = -K_0(g(X,Z)g(Y,T) - g(Y,Z)g(X,T)) \ .$$

(23) For a pseudo-Riemannian manifold with Riemann tensor R we define the **Ricci tensor** to be

$$P(X,Y) = -tr(Z \longmapsto R(X,Z)Y) \ ,$$

for all X and $Y \in \mathcal{X}(M)$. Here $tr(A)$ denotes the trace, i. e. the sum of the diagonal entries of any matrix representing the linear map A.

Definition 12. The **Ricci curvature** of an n-dimensional pseudo-Riemannian manifold (M,g) in the direction of $X \in \mathcal{X}(M)$ is

$$Ric(X) = \frac{1}{n-1} P(X,X) \ .$$

Clearly the Ricci curvature determines the Ricci tensor by polarization.

(24) We want to work out concrete formulae for the Ricci tensor and the Ricci curvature in terms of sectional curvatures. For this purpose we use a local orthonormal frame $X_1, \ldots, X_n$. Here orthonormal means that the X_i are pairwise orthogonal with respect to g, and that $g(X_i, X_i) = \pm 1$ for all i. (For a Riemannian manifold all these scalar products are $+1$.) With respect to such a local frame the trace of an endomorphism A of TM can be written as

$$tr(A) = \sum_{i=1}^{n} \left(g(AX_i, X_i) \cdot g(X_i, X_i) \right) \ .$$

Thus, we have

$$P(X,Y) = -\sum_{i=1}^{n} \left(g(R(X,X_i)Y, X_i) \cdot g(X_i, X_i) \right) \ ,$$

and

$$Ric(X) = \frac{1}{n-1} P(X,X) = -\frac{1}{n-1} \sum_{i=1}^{n} \left(g(R(X,X_i)X,X_i) \cdot g(X_i,X_i) \right)$$
$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(g(R(X,X_i)X_i,X) \cdot g(X_i,X_i) \right) \ .$$

If X is not a null vector, we may assume that

$$X_1 = \frac{1}{\sqrt{|g(X,X)|}} X \ ,$$

so that the above formula becomes

$$\begin{aligned} Ric(X) &= \frac{1}{n-1} \sum_{i=1}^{n} \left(g(R(X,X_i)X_i,X) \cdot g(X_i,X_i) \right) \\ &= |g(X,X)| \cdot \frac{1}{n-1} \sum_{i=2}^{n} \left(g(R(X_1,X_i)X_i,X_1) \cdot g(X_i,X_i) \right) \\ &= |g(X,X)| \cdot \frac{1}{n-1} \sum_{i=2}^{n} \left(K(Span\{X_1,X_i\}) \cdot g(X_1,X_1) \cdot g(X_i,X_i)^2 \right) \\ &= g(X,X) \cdot \frac{1}{n-1} \sum_{i=2}^{n} K(Span\{X_1,X_i\}) \ . \end{aligned}$$

The last line shows that the Ricci curvature is essentially an average of sectional curvatures.
(25) We now want to define a scalar measure of curvature s by taking the trace of the Ricci tensor with respect to the metric g. Because g is non-degenerate, there exists a unique $A \in \Gamma(End(TM))$ such that

$$P(X,Y) = g(AX,Y)$$

holds for all vector fields X and Y.

Definition 13. The **scalar curvature** of a pseudo-Riemannian manifold (M,g) is the function s$\in C^\infty(M)$ defined by $s = tr(A)$.

Using a local orthonormal frame as before, and substituting from the above formula expressing the Ricci curvature in terms of sectional curvatures, we have the following:

$$
\begin{aligned}
s = tr(A) &= \sum_{i=1}^{n} \left(g(AX_i, X_i) \cdot g(X_i, X_i) \right) \\
&= \sum_{i=1}^{n} \left(P(X_i, X_i) \cdot g(X_i, X_i) \right) \\
&= (n-1) \sum_{i=1}^{n} \left(Ric(X_i) \cdot g(X_i, X_i) \right) \\
&= (n-1) \sum_{i=1}^{n} \left(\left(g(X_i, X_i) \cdot \frac{1}{n-1} \sum_{j \neq i} K(Span\{X_i, X_j\}) \right) \cdot g(X_i, X_i) \right) \\
&= \sum_{i \neq j} K(Span\{X_i, X_j\}) \\
&= 2 \sum_{i<j} K(Span\{X_i, X_j\}) \ .
\end{aligned}
$$

(26) Here are some remarks and examples for curvature calculations. We say that a pseudo-Riemannian manifold is **flat** if its Riemann tensor vanishes identically. Equivalently, all sectional curvatures vanish.

- If M is $\mathbb{R}^n$ and g is given by a constant scalar product, then the coordinate vector fields are parallel, and the Levi-Cività connection is flat. So (M, g) is flat as a pseudo-Riemannian manifold.
- Suppose we scale a given metric g by replacing it with $\lambda^2 g$ for some non-zero real number λ. Then the Levi-Cività connection of g is also compatible with $\lambda^2 g$, and is therefore the Levi-Cività connection of this new metric. This implies that the Riemann tensor is unchanged. Looking at the definition of sectional curvature, we see that a two-plane σ is non-degenerate for $\lambda^2 g$ if and only if it is non-degenerate for g, and that the two sectional curvatures are related as follows:
$$K^{\lambda^2 g}(\sigma) = \frac{1}{\lambda^2} K^g(\sigma) \ .$$
The formula expressing the Ricci curvature in terms of sectional curvatures shows
$$Ric^{\lambda^2 g}(X) = Ric^g(X)$$
for all non-isotropic X. By continuity we have the same conclusion for all X. Finally, the formula expressing the scalar curvature in terms of sectional curvatures shows
$$s^{\lambda^2 g} = \frac{1}{\lambda^2} s^g \ .$$
- If M is two-dimensional and oriented and g is Riemannian, then the sectional curvature of T_pM equals the Gaussian curvature $\kappa(p)$, and the scalar curvature is twice the Gaussian curvature.

(27) We now consider the following situation: (N, g) is a pseudo-Riemannian manifold, and $M \subset N$ is a smooth hypersurface, for which we assume that $T_pM \subset T_pN$ is a non-degenerate subspace with respect to g for all $p \in M$. This assumption ensures that $T_pN = T_pM \oplus (T_pM)^\perp$ for all $p \in M$, where $(T_pM)^\perp$ is the orthogonal space with respect to g. (This is not just a direct sum, but also an orthogonal sum with respect to g.) As M is a hypersurface it follows that $(T_pM)^\perp$ is one-dimensional, spanned by a vector n. By non-degeneracy this vector cannot be null, so that by scaling we may assume $g(n, n) = \pm 1$.

The non-degeneracy of the subspace $T_pM \subset T_pN$ for all $p \in M$ means that the restriction of g to TM is a pseudo-Riemannian metric h on M. The Levi-Cività connection ∇^M of (M, h) can be obtained from the Levi-Cività connection ∇ of (N, g) as follows:

$$\nabla^M_X Y = \pi \nabla_{\tilde{X}} \tilde{Y} ,$$

where $\tilde{X}$ and $\tilde{Y}$ are local extensions of $X, Y \in \mathcal{X}(M)$ to N and $\pi \colon TN|_M \to TM$ is the projection with kernel $(T_pM)^\perp$. As this is independent of the choices of local extensions for X and Y, we will not specify the extensions in later formulae.

(28) Suppose we have chosen a normal vector field n for $M \subset N$, normalized so that $g(n, n) = \pm 1$.

Definition 14. The **Weingarten map** at $p \in M$ is the linear map

$$\begin{aligned} L\colon T_pM &\longrightarrow T_pM \\ X &\longmapsto \nabla_X n . \end{aligned}$$

Again we need to choose a local extension of n in a neighborhood of p, but the result we get is independent of the choice of extension. It follows from the constancy of $g(n, n)$ that $\nabla_X n$ is orthogonal to n and therefore contained in T_pM. Note that the normalization determines n uniquely up to sign. If we replace n by $-n$, then we obtain $-L$ instead of L.

The Weingaten map has the following easily proved property, see [2], Section 10.3.

Lemma 15. *The Weingarten map is symmetric with respect to the metric, i. e.* $h(L(X), Y) = (X, L(Y))$ *for all* $X, Y \in T_pM$.

(29) For $X, Y, Z \in T_pM$ we can relate the value of $R(X, Y)Z$, the curvature of (N, g), and of $R^M(X, Y)Z$, the curvature of (M, h), through the Weingarten map. In the special case that the Levi-Cività connection ∇ of (N, g) is flat we obtain:

Theorem 16 (Gauss Equation)**.** *If the pseudo-Riemannian manifold* (N, g) *is flat, and* $M \subset N$ *is a hypersurface for which the restriction* h *of* g *to* M *is also pseudo-Riemannian, then the curvature of the Levi-Cività connection of* (M, h) *is given by*

$$R^M(X, Y)Z = g(n, n) \left(h(L(Y), Z)L(X) - h(L(X), Z)L(Y) \right) ,$$

where n *is a normal vector to* M *normalized so that* $g(n, n) = \pm 1$.

Notice that this formula is unchanged if we replace n by $-n$.

(30) As an example for the above situation we take $N = \mathbb{R}^{n+1}$, with g given by the standard positive definite scalar product. Note that because g is positive definite, all subspaces are positive definite as well, and are therefore non-degenerate.

For the hypersurface M we take the unit sphere $S^n \subset \mathbb{R}^{n+1}$. As normal vector field n we take the outer unit normal to the sphere, which at $p \in S^n$ is just p itself. Now we can calculate that in this case the Weingarten map L is the identity, so that the curvature tensor of the unit sphere endowed with the induced Riemannian metric h is given by

$$R^{S^n}(X,Y)Z = h(Y,Z)X - h(X,Z)Y \ .$$

Comparing this with Proposition 11 we see that (S^n, h) has constant sectional curvature equal to $+1$.

(31) A property (potentially) weaker than that of having constant sectional curvature is to have constant Ricci curvaure:

Definition 17. A pseudo-Riemannian manifold (M, g) is said to be **Einstein** if there is a real constant μ such that $P(X,Y) = \mu g(X,Y)$ for all X and Y.

By polarization this is equivalent to the existence of a constant $\lambda \in \mathbb{R}$ such that $Ric(X) = \lambda g(X,X)$ for all X.

Note that constant sectional curvature implies the Einstein condition, which in turn implies that the scalar curvature is constant. If (M, g) is two-dimensional and Riemannian, then all these conditions are in fact equivalent, and are equivalent to constancy of the Gaussian curvature.

References

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