Geometry of manifolds II
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(1) We begin with some standard material on Lie groups and their Lie algebras, using mainly [1], Chapter 5, as our reference. See [2], Chapter 3, for a slightly different treatment.

A **Lie group** G is a smooth manifold with a group structure for which inversion and multiplication are smooth maps. Discussion of basic properties and examples.

(2) For every element g of a Lie group G we can consider left multiplication l_g, also called left translation, by g:

$$\begin{aligned} l_g \colon G &\longrightarrow G \\ a &\longmapsto g \cdot a \ . \end{aligned}$$

Right translation r_g is defined analogously, and is usually different from left translation because G need not be commutative. Left and right translations are diffeomorphisms of G.

(3) A **Lie algebra** V (over $\mathbb{R}$) is a vector space (over $\mathbb{R}$) together with a bilinear map $[\ ,\ ] \colon V \times V \to V$ which is skew-symmetric and satisfies the Jacobi identity

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \ .$$

(4) A vector field X on a Lie group G is called left-invariant if it is invariant under left translation, i. e. $(D_a l_g)(X(a)) = X(g \cdot a)$ for all $g, a \in G$.

Proposition 1. *The subset of left-invariant vector fields in $\mathcal{X}(G)$ is a linear subspace which is closed under commutation, and is therefore a Lie algebra.*

Definition 2. The space $\mathfrak{g}$ of left-invariant vector fields on G is called the Lie algebra of G.

(5) Consider the evaluation map

$$\begin{aligned} ev \colon \mathfrak{g} &\longrightarrow T_e G \\ X &\longmapsto X(e) \end{aligned}$$

which assigns to each left-invariant vector field its value at the neutral element.

Proposition 3. *The evaluation map is a linear isomorphism of vector spaces.*

This result holds for the evaluation map $\mathfrak{g} \to T_g G$ at any $g \in G$. We shall often implicitely identify $\mathfrak{g}$ with $T_e G$, giving the tangent space at the neutral element the Lie algebra structure induced from that of $\mathfrak{g}$ via the evaluation isomorphism.

Corollary 4. *Any Lie group G has a well-defined dimension as a manifold (in other words, all its connected components have the same dimension), and this coincides with the dimension of its Lie algebra $\mathfrak{g}$ as a vector space over $\mathbb{R}$.*

Proof. Left translation by g is a diffeomorphism, whose inverse is left translation by g^{-1}. This shows that all components of G have the same dimension as manifolds. The dimension of the identity component is the dimension of $T_e G \cong \mathfrak{g}$ as a vector space. $\square$

Corollary 5. *Every Lie group G has trivial tangent bundle. An explicit trivialization is given by*

$$\begin{aligned} G \times \mathfrak{g} &\longrightarrow TG \\ (g, X) &\longmapsto X(g) \ . \end{aligned}$$

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(6) We now begin the discussion of homomorphisms between Lie groups. As a first special case consider the following:

Definition 6. A 1**-parameter subgroup** of a Lie group G is a smooth map $s\colon \mathbb{R} \to G$ with the properties $s(0) = e$ and $s(t_1 + t_2) = s(t_1) + s(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

Thus a 1-parameter subgroup is a smooth homomorphism from the Lie group $(\mathbb{R}, +)$ to G.

Proposition 7. *Let G be a Lie group, and $X \in \mathfrak{g}$. Then*

(a) *X is complete, i. e. it generates a global flow.*

(b) *There is a unique 1-parameter subgroup $s_X\colon \mathbb{R} \to G$ such that $\dot{s}_X(0) = X(e)$. The flow of X is given by $\phi_t(g) = g \cdot s_X(t)$.*

(7) In the case when G is $GL(n,\mathbb{R})$ we can identify T_eG, and therefore $\mathfrak{g}$, with $Mat(n \times n,\mathbb{R})$. The 1-parameter subgroup corresponding to some $X \in Mat(n \times n,\mathbb{R})$ is given by $s_X(t) = exp(tX)$, where the exponential is calculated using the matrix X in the usual series expansion. This motivates the following:

Definition 8. For any Lie group G, its **exponential map** is defined by

$$\begin{aligned} exp\colon \mathfrak{g} &\longrightarrow G \\ X &\longmapsto s_X(1)\,. \end{aligned}$$

This is well-defined because every left-invariant vector field is complete. The following is immediate:

Lemma 9. *For all $t \in \mathbb{R}$ we have $s_X(t) = exp(tX)$.*

(8) As in the case of the exponential map of a Riemannian manifold, defined using unit-speed geodesics with given initial values, we have:

Proposition 10. *The exponential map is smooth, and its derivative at the origin is the identity map of T_eG.*

Here again we have identified $\mathfrak{g}$ with T_eG. As an immediate consequence of the above, the inverse function theorem tells us that the exponential map is a local diffeomorphism.

(9) In general, a **homomorphism** of Lie groups is a smooth map which is also a homomorphism.

Proposition 11. *A homomorphism of Lie groups $f\colon G \to H$ induces a homomorphism of Lie algebras $f_*\colon \mathfrak{g} \to \mathfrak{h}$.*

Proof. The map f_* is defined by the composition

$$\mathfrak{g} \xrightarrow{ev} T_eG \xrightarrow{D_ef} T_eH \xrightarrow{ev^{-1}} \mathfrak{h}\,.$$

This is clearly a linear map of vector spaces. Thus it remains to prove

(*) $$f_*[X,Y] = [f_*X, f_*Y]\,.$$

We shall do this in the next lecture. □

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(10) We still have to prove (*). To do so, we first prove:

Lemma 12. *If $f\colon G \to H$ is a homomorphism of Lie groups and $X \in \mathfrak{g}$, then $(D_g f)(X(g)) = (f_* X)(f(g))$.*

After this, the fact that we are dealing with a homomorphism of Lie groups plays no further role in the proof. We prove the following:

Lemma 13. *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let $X, Y \in \mathcal{X}(M)$ and $\bar{X}, \bar{Y} \in \mathcal{X}(N)$ be vector fields with the properties $(D_p f)(X(p)) = \bar{X}(f(p))$ and $(D_p f)(Y(p)) = \bar{Y}(f(p))$. Then $(D_p f)([X,Y](p)) = [\bar{X}, \bar{Y}](f(p))$.*

Applying this to a Lie group homomorphism, Lemma 12 shows that we may set $\bar{X} = f_* X$ and $\bar{Y} = f_* Y$ to complete the proof of (*).

(11) A subset $H \subset G$ is said to be a **Lie subgroup** if it can be given the structure of a Lie group in such a way that the inclusion map $i\colon H \to G$ is a homomorphism and an injective immersion. Note that i is not required to be an embedding, because the subspace topology on H may not agree with its abstract Lie group topology.

Theorem 14. *There is a bijection between connected Lie subgroups $H \subset G$ of G and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{g}$.*

In one direction, if $H \subset G$ is a Lie subgroup, then $i_*\colon \mathfrak{h} \to \mathfrak{g}$ is injective and the image is a Lie subalgebra of $\mathfrak{g}$.

(12) Before proceeding with the proof of Theorem 14, we need to prepare some facts concerning subbundles of the tangent bundle of a smooth manifold. See Chapter 4 of [1] or pages 41–49 of [2].

Definition 15. A **distribution** $\mathcal{D}$ of rank k on a smooth manifold M is a smooth subbundle $\mathcal{D} \subset TM$ or rank k.

The commutator of vector fields with values in $\mathcal{D}$ defines a bilinear map $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \mathcal{X}(M)$, which usually does not take values in $\Gamma(\mathcal{D})$ only. The cases when when it does are characterized by the following result:

Theorem 16 (Frobenius Theorem)**.** *For a rank k distribution $\mathcal{D} \subset TM$ the following two conditions are equivalent:*

(a) *the space of sections $\Gamma(\mathcal{D})$ is closed under $[\ ,\ ]$, and*
(b) *for every $p \in M$ there is a smooth manifold L_p of dimension k and an injective immersion $i\colon L_p \to M$ with $p \in i(L_p)$ and $(D_q i)(T_q L_p) = \mathcal{D}(i(q))$ for all $q \in L_p$.*

That (b) implies (a) is easy to see, essentially using Lemma 13. The converse is more interesting. One chooses a local frame for $\mathcal{D}$ around a point $p \in M$, and argues that this can be arranged to consist of commuting vector fields using (a). Then the local flows of these (local) vector fields also commute, and moving p by these commuting flows produces a local submanifold through the point p, which is everywhere tangent to $\mathcal{D}$.

The manifolds L_p are called **integral manifolds** of $\mathcal{D}$, and a distribution $\mathcal{D}$ satisfying the conditions of the Frobenius theorem is said to be (completely) integrable. One can define an equivalence relation on the points of M by defining two points to be equivalent if they are contained in a connected integral manifold for $\mathcal{D}$. This defines a foliation on M, whose leaves are the equivalence classes of this equivalence relation, i. e. the maximal connected integral manifolds.

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(13) We now return to the proof of Theorem 14. Suppose we are given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. By evaluation at all points of G this defines a left-invariant distribution $\mathcal{D}$ on G. The assumption that $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, i. e. it is closed under commutation, implies that $\mathcal{D}$ satisfies condition (a) in the Frobenius theorem. By the Frobenius theorem $\mathcal{D}$ is completely integrable. Let H be the leaf of the corresponding foliation through the neutral element $e \in G$. The leaf through $a \in G$ is then obtained by left translation with a applied to H. It turns out that H is then a Lie subgroup, and is the only connected such group with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. This completes the proof of Theorem 14.

(14) Lie subgroups are not usually closed subsets. The easiest example of this phenomenon is a densely immersed copy of $\mathbb{R}$ in T^2.

Proposition 17. *If the immersion of a Lie subgroup $H \subset G$ is proper, then H is a closed subset of G.*

(15) The above proposition has a strong converse:

Theorem 18. *If G is a Lie group and $H \subset G$ is a closed subset that is also an abstract subgroup, then H is a Lie subgroup whose inclusion is proper.*

Note that H is only assumed to be an abstract subgroup, not a Lie subgroup. For the proof, see [1], p. 139–141.

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(16) We now begin the discussion of Lie group actions on manifolds. All our actions will be smooth:

Definition 19. A (left) action of a Lie group G on a smooth manifold M is a smooth map $\mu \colon G \times M \to M$ such that

- $\mu(e, p) = p$ for all $p \in M$, and
- $\mu(g_1, \mu(g_2, p)) = \mu(g_1 g_2, p)$ for all $p \in M$, $g_1, g_2 \in G$.

When it is clear which action of G on M we have in mind, we often simplify the notation and write $g(p)$ or $g \cdot p$ instead of $\mu(g, p)$. Then the second defining property can be expressed as $g_1(g_2(p)) = (g_1 g_2)(p)$.

Remark 20. When nothing further is said, all our actions will be left actions. Right actions are defined analogously, so that $e(p) = p$ and $g_1(g_2(p)) = (g_2 g_1)(p)$.

An action is said to be **effective** if for all $g \neq e \in G$ there exists a $p \in M$ such that $g(p) \neq p$. An action is **transitive** if for all $p, q \in M$ there exists a $g \in G$ such that $g(p) = q$. The **orbit** of a point $p \in M$ under the action consists of all the points $g(p)$. Partitioning M into the orbits of the action defines an equivalence relation on M. Transitivity is equivalent to the requirement that there is only one orbit.

(17) Examples: the trivial action, the action of $\mathbb{R}$ defined by a flow on M, the action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$, the action of $SO(n)$ on $\mathbb{R}^n$, its restriction to S^{n-1},...

(18) For every $p \in M$ we define $G_p \subset G$ to consist of the elements $g \in G$ with the property $g(p) = p$. This is a subgroup of G called the **isotropy subgroup** of p.

Lemma 21. *For every $p \in M$ the isotropy subgroup G_p is a closed subgroup of G.*

Note that Theorem 18 now tells us that G_p is a properly embedded Lie subgroup of G.

(19) Let G be a Lie group, and $H \subset G$ a closed subgroup. The set G/H of left cosets of H in G is a topological space with the quotient topology, and is called a **homogeneous space**.

Theorem 22. *For every closed subgroup $H \subset G$ the corresponding homogeneous space G/H has a natural structure as a smooth manifold of dimension $dim(G) - dim(H)$, such that the projection $\pi\colon G \to G/H$ is smooth and admits smooth local sections.*

In fact the projection will turn out to be a submersion. In the proof we shall use the notation $l_a\colon G/H \to G/H$ for the continuous maps induced on G/H by the left translations on G. These are all homeomorphisms, and will turn into diffeomorphisms once we have defined the smooth structure on G/H.

Proof. By definition, a subset $U \subset G/H$ is open if and only if its preimage $\pi^{-1}(U) \subset G$ is open. Therefore, a countable basis for the topology of G induces a countable basis for the topology of G/H.

The assumption that $H \subset G$ is a closed subset implies that its preimage $R \subset G \times G$ under the map $(g_1, g_2) \mapsto g_1^{-1} g_2$ is also closed. If $aH \neq bH$, then $(a, b) \notin R$, so we can find open neighbourhoods U and V of a respectively b in G so that $(U \times V) \cap R = \emptyset$. Then $\pi(U)$ and $\pi(V)$ are disjoint open neighbourhoods of aH respectively bH in G/H. Thus G/H has the Hausdorff property.

It remains to construct an atlas with smooth transition maps. First, let (V, ϕ) be a chart for G around e. As $H \subset G$ is closed, we may assume that $H \cap V$ is connected. We choose ϕ so that it maps the intersections of the cosets of H with V to the subsets $\mathbb{R}^k \times \{c\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n = \phi(V)$. By shrinking V, we find smaller open sets U and V_1 which have the same properties and in addition satisfy $U^{-1} \cdot U \subset V_1$ and $V_1 \cdot V_1 \subset V$. It follows that every coset of H either does not meet V_1 at all, or meets it in a connected subset diffeomorphic via ϕ to $\mathbb{R}^k$.

Define $\tilde{\phi}^{-1}\colon \mathbb{R}^{n-k} \to \pi(U)$ by $\pi \circ \phi^{-1}$, where $\mathbb{R}^{n-k} = \{0\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$. This map is bijective, continuous and an open map, and so is a homeomorphism. We define $\tilde{\phi}\colon \pi(U) \to \mathbb{R}^{n-k}$ by inversion, and this is then also a homeomorphism. We shall use this as a chart for G/H around the point $eH = H$.

For any $a \in G$ we obtain a chart for G/H around aH by taking $(\pi(l_a(U)), \tilde{\phi}_{aH} = \tilde{\phi} \circ l_{a^{-1}})$. Thus, we have the structure of a topological manifold on G/H. To see that this atlas actually defines a smooth structure, it remains to check that the transition maps between different charts are smooth. Suppose that $\pi(l_a(U)) \cap \pi(l_b(U)) \neq \emptyset$. Then we have to prove that $\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1}$ is smooth on $X = \tilde{\phi}_{aH}(\pi(l_a(U)) \cap \pi(l_b(U)))$. Note that

$$\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1} = (\tilde{\phi} \circ l_{b^{-1}}) \circ (\tilde{\phi} \circ l_{a^{-1}})^{-1} = \tilde{\phi} \circ l_{b^{-1}a} \circ \tilde{\phi}^{-1} .$$

Let $p \in X$. Then $l_{b^{-1}a} \circ \tilde{\phi}^{-1}(p) \in \pi(U)$, so there exists a $g \in H$ such that $l_{b^{-1}a} \circ \phi^{-1}(p) \cdot g \in U$. There is a neighbourhood W of p in X such that $l_{b^{-1}a} \circ \phi^{-1}(W) \cdot g \subset U$. On W we can rewrite $\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1}$ as

$$\pi_2 \circ \phi \circ r_g \circ l_{b^{-1}a} \circ \phi^{-1} ,$$

where π_2 is the projection from $\mathbb{R}^k \times \mathbb{R}^{n-k}$ to the second factor. This composition is clearly smooth.

Thus, we have finally checked that G/H is a smooth manifold. In the charts we have constructed π corresponds to π_2 and is therefore smooth. The smooth local sections are provided by the left translations of the restriction of ϕ^{-1} to the second factor in $\mathbb{R}^k \times \mathbb{R}^{n-k}$. $\square$

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.
2. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag 1983.