Mathematical gauge theory I

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(1) This is a course on the geometry of principal bundles, connections, curvature and gauge transformations. All this material is used in physics for the discussions of classical gauge theories such as electromagnetism and QCD.

(2) You are assumed to be familiar with smooth manifolds and smooth maps, with tangent bundles, vector fields and their flows, and with Lie derivatives.

(3) We begin by recalling some standard material on Lie groups and their Lie algebras, using mainly [1], Chapter 5, and [2], Chapter 3, as references.

A **Lie group** G is a smooth manifold with a group structure for which inversion and multiplication are smooth maps. Discussion of basic properties and examples.

(4) For every element g of a Lie group G we can consider left multiplication l_g, also called left translation, by g:

$$\begin{aligned} l_g \colon G &\longrightarrow G \\ a &\longmapsto g \cdot a \, . \end{aligned}$$

Right translation r_g is defined analogously, and is usually different from left translation because G need not be commutative. Left and right translations are diffeomorphisms of G.

(5) A **Lie algebra** V (over $\mathbb{R}$) is a vector space (over $\mathbb{R}$) together with a bilinear map $[\ ,\ ] \colon V \times V \to V$ which is skew-symmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \, .$$

(6) A vector field X on a Lie group G is called left-invariant if it is invariant under left translation, i. e. $(D_a l_g)(X(a)) = X(g \cdot a)$ for all $g, a \in G$.

Proposition 1. *The subset of left-invariant vector fields in $\mathcal{X}(G)$ is a linear subspace which is closed under commutation, and is therefore a Lie algebra.*

Definition 2. The space $\mathfrak{g}$ of left-invariant vector fields on G is called the Lie algebra of G.

(7) Consider the evaluation map

$$\begin{aligned} ev \colon \mathfrak{g} &\longrightarrow T_e G \\ X &\longmapsto X(e) \end{aligned}$$

which assigns to each left-invariant vector field its value at the neutral element.

Proposition 3. *The evaluation map is a linear isomorphism of vector spaces.*

This result holds for the evaluation map $\mathfrak{g} \to T_g G$ at any $g \in G$. We shall often implicitly identify $\mathfrak{g}$ with $T_e G$, giving the tangent space at the neutral element the Lie algebra structure induced from that of $\mathfrak{g}$ via the evaluation isomorphism.

19 April 2012

(8) The Proposition proved at the end of the first lecture has some important consequences.

Corollary 4. *Any Lie group G has a well-defined dimension as a manifold (in other words, all its connected components have the same dimension), and this coincides with the dimension of its Lie algebra $\mathfrak{g}$ as a vector space over $\mathbb{R}$.*

Proof. Left translation by g is a diffeomorphism, whose inverse is left translation by g^{-1}. This shows that all components of G have the same dimension as manifolds. The dimension of the identity component is the dimension of $T_eG \cong \mathfrak{g}$ as a vector space. □

Corollary 5. *Every Lie group G has trivial tangent bundle. An explicit trivialization is given by*

$$\begin{aligned} G \times \mathfrak{g} &\longrightarrow TG \\ (g, X) &\longmapsto X(g) \ . \end{aligned}$$

(9) Next we discuss homomorphisms between Lie groups. As a first special case consider the following:

Definition 6. A 1-**parameter subgroup** of a Lie group G is a smooth map $s\colon \mathbb{R} \to G$ with the properties $s(0) = e$ and $s(t_1 + t_2) = s(t_1) + s(t_2)$ for all $t_1, t_2 \in \mathbb{R}$.

Thus a 1-parameter subgroup is a smooth homomorphism from the Lie group $(\mathbb{R}, +)$ to G.

Proposition 7. *Let G be a Lie group, and $X \in \mathfrak{g}$. Then*
(a) *X is complete, i. e. it generates a global flow.*
(b) *There is a unique 1-parameter subgroup $s_X\colon \mathbb{R} \to G$ such that $\dot{s}_X(0) = X(e)$. The flow of X is given by $\phi_t(g) = g \cdot s_X(t)$.*

(10) In the case when G is $GL(n, \mathbb{R})$ we can identify T_eG, and therefore $\mathfrak{g}$, with $Mat(n \times n, \mathbb{R})$. The 1-parameter subgroup corresponding to some $X \in Mat(n \times n, \mathbb{R})$ is given by $s_X(t) = exp(tX)$, where the exponential is calculated using the matrix X in the usual series expansion. This motivates the following:

Definition 8. For any Lie group G, its **exponential map** is defined by

$$\begin{aligned} exp\colon \mathfrak{g} &\longrightarrow G \\ X &\longmapsto s_X(1) \ . \end{aligned}$$

This is well-defined because every left-invariant vector field is complete. The following is immediate:

Lemma 9. *For all $t \in \mathbb{R}$ we have $s_X(t) = exp(tX)$.*

(11) As in the case of the exponential map of a Riemannian manifold, defined using unit-speed geodesics with given initial values, we have:

Proposition 10. *The exponential map is smooth, and its derivative at the origin is the identity map of T_eG.*

Here again we have identified $\mathfrak{g}$ with T_eG. As an immediate consequence of the above, the inverse function theorem tells us that the exponential map is a local diffeomorphism. The proof of this Proposition is left as an exercise on the first homework assignment.

(12) In general, a **homomorphism** of Lie groups is a smooth map which is also a homomorphism.

Proposition 11. *A homomorphism of Lie groups $f\colon G \to H$ induces a homomorphism of Lie algebras $f_*\colon \mathfrak{g} \to \mathfrak{h}$.*

Proof. The map f_* is defined by the composition

$$\mathfrak{g} \xrightarrow{ev} T_eG \xrightarrow{D_ef} T_eH \xrightarrow{ev^{-1}} \mathfrak{h} \ .$$

This is clearly a linear map of vector spaces. Thus it remains to prove

(*) $$f_*[X,Y] = [f_*X, f_*Y] \ .$$

This is done in the next two lemmas. □

Lemma 12. *If $f\colon G \to H$ is a homomorphism of Lie groups and $X \in \mathfrak{g}$, then $(D_gf)(X(g)) = (f_*X)(f(g))$.*

After this, the fact that we are dealing with a homomorphism of Lie groups plays no further role in the proof.

23 April 2012

(13) We prove the following:

Lemma 13. *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let $X, Y \in \mathcal{X}(M)$ and $\bar{X}, \bar{Y} \in \mathcal{X}(N)$ be vector fields with the properties $(D_pf)(X(p)) = \bar{X}(f(p))$ and $(D_pf)(Y(p)) = \bar{Y}(f(p))$. Then $(D_pf)([X,Y](p)) = [\bar{X},\bar{Y}](f(p))$.*

Applying this to a Lie group homomorphism, Lemma 12 shows that we may set $\bar{X} = f_*X$ and $\bar{Y} = f_*Y$ to complete the proof of (*).

(14) Before proceeding with the discussion of Lie subgroups, we need to prepare some facts concerning subbundles of the tangent bundle of a smooth manifold. These will also be useful later on for the discussion of flat connections on principal bundles. We refer ro Chapter 4 of [1] or pages 41–49 of [2] for this material on integrability.

Theorem 14. *Let $X_1, \dots, X_k \in \mathcal{X}(M)$ be vector fields with $[X_i, X_j] = 0$ for all i and j. If $X_1(p), \dots, X_k(p)$ are linearly independent in T_pM, then there is a chart (U, φ) with $p \in U$ such that $D\varphi(X_i|_U) = \frac{\partial}{\partial x_i}$ for all $i = 1, \dots, k$.*

(15) A **distribution** E of rank k on a smooth manifold M is a smooth subbundle $E \subset TM$ of rank k. The commutator of vector fields with values in E defines a bilinear map $\Gamma(E) \times \Gamma(E) \to \mathcal{X}(M)$, which usually does not take values in $\Gamma(E)$ only.

Definition 15. A distribution E is involutive, if $\Gamma(E)$ is closed under $[\ ,\ ]$.

Definition 16. A distribution E of rank k is integrable, if through every point $p \in M$ there is a submanifold N with $TN = E|_N$. Such an N is called an integral manifold for E.

Theorem 17 (Frobenius Theorem)**.** *For a rank k distribution $E \subset TM$ the following conditions are equivalent:*

(1) *E is integrable,*
(2) *E is involutive,*
(3) *there is a covering of M by domains of charts (U, φ) such that $D\varphi(E|_U)$ contains $\frac{\partial}{\partial x_i}$ for all $i = 1, \ldots, k$.*

That (3) implies (1) is easy to see. Further, (1) implies (2) by Lemma 13. Finally, the implication from (2) to (3) is the interesting part. One chooses a local frame for E around a point $p \in M$, and argues that this can be arranged to consist of commuting vector fields using (2). Then the local flows of these (local) vector fields also commute, and one can use Theorem 14.

26 April 2012

(16) A subset $H \subset G$ is said to be a **Lie subgroup** if it can be given the structure of a Lie group in such a way that the inclusion map $i\colon H \to G$ is a homomorphism and an injective immersion. Note that i is not required to be an embedding, because the subspace topology on H may not agree with its abstract Lie group topology.

Theorem 18. *There is a bijection between connected Lie subgroups $H \subset G$ of G and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{g}$.*

In one direction, if $H \subset G$ is a Lie subgroup, then $i_*\colon \mathfrak{h} \to \mathfrak{g}$ is injective and the image is a Lie subalgebra of $\mathfrak{g}$.

For the converse suppose we are given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. By evaluation at all points of G this defines a left-invariant distribution E on G. The assumption that $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, i. e. it is closed under commutation, implies that E is involutive. By the Frobenius theorem E is integrable. Let H be the leaf of the corresponding foliation through the neutral element $e \in G$. The leaf through $a \in G$ is then obtained by left translation with a applied to H. It turns out that H is then a Lie subgroup, and is the only connected such group with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. This completes the proof of Theorem 18.

(17) Lie subgroups are not usually closed subsets. The easiest example of this phenomenon is a densely immersed copy of $\mathbb{R}$ in T^2.

(18) We state the following theorem without giving the proof. For the proof, see [1], p. 139–141.

Theorem 19. *If G is a Lie group and $H \subset G$ is a closed subset that is also an abstract subgroup, then H is a Lie subgroup.*

(19) We now begin the discussion of Lie group actions on manifolds. All our actions will be smooth:

Definition 20. A (left) action of a Lie group G on a smooth manifold M is a smooth map $\mu\colon G \times M \to M$ such that

- $\mu(e, p) = p$ for all $p \in M$, and
- $\mu(g_1, \mu(g_2, p)) = \mu(g_1 g_2, p)$ for all $p \in M$, $g_1, g_2 \in G$.

When it is clear which action of G on M we have in mind, we often simplify the notation and write $g(p)$ or $g{\cdot}p$ instead of $\mu(g, p)$. Then the second defining property can be expressed as $g_1(g_2(p)) = (g_1 g_2)(p)$.

(20) Right actions are defined analogously, so that $e(p) = p$ and $g_1(g_2(p)) = (g_2g_1)(p)$. One can convert left and right actions into each other, but it is sometimes important to distinguish them.
(21) An action is said to be **effective** if for all $g \neq e \in G$ there exists a $p \in M$ such that $g(p) \neq p$. An action is **transitive** if for all $p, q \in M$ there exists a $g \in G$ such that $g(p) = q$. The **orbit** of a point $p \in M$ under the action consists of all the points $g(p)$. Partitioning M into the orbits of the action defines an equivalence relation on M. Transitivity is equivalent to the requirement that there is only one orbit.

30 April 2012

(22) Let $f\colon G \longrightarrow H$ be a homomorphism of Lie groups. Then it induces the homomorphism $f_*\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ between the Lie algebras, and this satisfies $exp \circ f_* = f \circ exp$.
(23) Whenever an action of G on M fixes a point $p \in M$, we can take derivatives at p to obtain a representation $G \longrightarrow GL(T_pM)$. This is called the **isotropy representation** at p.
(24) Conjugation defines a left action of a Lie group G on itself. This action fixes the neutral element $e \in G$, and the isotropy representation at e is called the **adjoint representation** $Ad\colon G \longrightarrow GL(\mathfrak{g})$, where we identify $\mathfrak{g}$ with T_eG via the evaluation map using Proposition 3.
(25) Ad is a Lie group homomorphism inducing a map Ad_* between Lie algebras. This is denoted $ad\colon \mathfrak{g} \longrightarrow End(\mathfrak{g})$, and is also called the **adjoint representation**, but this is a representation of $\mathfrak{g}$, not of G.
(26) Item (22) above gives us various commutative diagrams intertwining Ad and ad, see [2, p. 114]. In the special case where $G = GL(V)$ for some vector space V, we have $Ad_g(M) = gMg^{-1}$.
(27) For all G we have $ad_X(Y) = L_XY = [X, Y]$.

3 May 2012

(28) Definition of the center of G and of $\mathfrak{g}$.
(29) The center of a connected Lie group G is precisely the kernel of Ad.
(30) Let G be a Lie group, and $H \subset G$ a closed subgroup. The set G/H of left cosets of H in G is a topological space with the quotient topology, and is called a **homogeneous space**.

Theorem 21. *For every closed Lie subgroup $H \subset G$ the corresponding homogeneous space G/H has a natural structure as a smooth manifold of dimension $dim(G) - dim(H)$, such that the projection $\pi\colon G \to G/H$ is smooth and admits smooth local sections.*

In fact the projection will turn out to be a submersion. In the proof we shall use the notation $l_a\colon G/H \to G/H$ for the continuous maps induced on G/H by the left translations on G. These are all homeomorphisms, and will turn into diffeomorphisms once we have defined the smooth structure on G/H.

Proof. By definition, a subset $U \subset G/H$ is open if and only if its preimage $\pi^{-1}(U) \subset G$ is open. Therefore, a countable basis for the topology of G induces a countable basis for the topology of G/H.

The assumption that $H \subset G$ is a closed subset implies that its preimage $R \subset G \times G$ under the map $(g_1, g_2) \mapsto g_1^{-1}g_2$ is also closed. If $aH \neq bH$, then $(a, b) \notin R$, so we can find open neighbourhoods U and V of a respectively b in G so that $(U \times V) \cap R = \emptyset$. Then $\pi(U)$ and $\pi(V)$ are disjoint open neighbourhoods of aH respectively bH in G/H. Thus G/H has the Hausdorff property.

It remains to construct an atlas with smooth transition maps. First, let (V, ϕ) be a chart for G around e. As $H \subset G$ is closed, we may assume that $H \cap V$ is connected. We choose ϕ so that it maps the intersections of the cosets of H with V to the subsets $\mathbb{R}^k \times \{c\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n = \phi(V)$. By shrinking V, we find smaller open sets U and V_1 which have the same properties and in addition satisfy $U^{-1} \cdot U \subset V_1$ and $V_1 \cdot V_1 \subset V$. It follows that every coset of H either does not meet V_1 at all, or meets it in a connected subset diffeomorphic via ϕ to $\mathbb{R}^k$.

Define $\tilde{\phi}^{-1} \colon \mathbb{R}^{n-k} \to \pi(U)$ by $\pi \circ \phi^{-1}$, where $\mathbb{R}^{n-k} = \{0\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$. This map is bijective, continuous and an open map, and so is a homeomorphism. We define $\tilde{\phi} \colon \pi(U) \to \mathbb{R}^{n-k}$ by inversion, and this is then also a homeomorphism. We shall use this as a chart for G/H around the point $eH = H$.

For any $a \in G$ we obtain a chart for G/H around aH by taking $(\pi(l_a(U)), \tilde{\phi}_{aH} = \tilde{\phi} \circ l_{a^{-1}})$. Thus, we have the structure of a topological manifold on G/H. To see that this atlas actually defines a smooth structure, it remains to check that the transition maps between different charts are smooth. Suppose that $\pi(l_a(U)) \cap \pi(l_b(U)) \neq \emptyset$. Then we have to prove that $\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1}$ is smooth on $X = \tilde{\phi}_{aH}(\pi(l_a(U)) \cap \pi(l_b(U)))$. Note that

$$\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1} = (\tilde{\phi} \circ l_{b^{-1}}) \circ (\tilde{\phi} \circ l_{a^{-1}})^{-1} = \tilde{\phi} \circ l_{b^{-1}a} \circ \tilde{\phi}^{-1} \,.$$

Let $p \in X$. Then $l_{b^{-1}a} \circ \tilde{\phi}^{-1}(p) \in \pi(U)$, so there exists a $g \in H$ such that $l_{b^{-1}a} \circ \phi^{-1}(p) \cdot g \in U$. There is a neighbourhood W of p in X such that $l_{b^{-1}a} \circ \phi^{-1}(W) \cdot g \subset U$. On W we can rewrite $\tilde{\phi}_{bH} \circ \tilde{\phi}_{aH}^{-1}$ as

$$\pi_2 \circ \phi \circ r_g \circ l_{b^{-1}a} \circ \phi^{-1} \,,$$

where π_2 is the projection from $\mathbb{R}^k \times \mathbb{R}^{n-k}$ to the second factor. This composition is clearly smooth.

Thus, we have finally checked that G/H is a smooth manifold. In the charts we have constructed π corresponds to π_2 and is therefore smooth. The smooth local sections are provided by the left translations of the restriction of ϕ^{-1} to the second factor in $\mathbb{R}^k \times \mathbb{R}^{n-k}$. □

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.
2. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag 1983.