LINKING NUMBERS OF MEASURED FOLIATIONS

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Abstract. We generalise the average asymptotic linking number of a pair of divergence-free vector fields on homology three-spheres [1, 2, 14] by considering the linking of a divergence-free vector field on a manifold of arbitrary dimension with a codimension two foliation endowed with an invariant transverse measure. We prove that the average asymptotic linking number is given by an integral of Hopf type. Considering appropriate vector fields and measured foliations, we obtain an ergodic interpretation of the Godbillon-Vey invariant of a family of codimension one foliations discussed in [9].

1. Introduction

Generalising the differential form description of the classical Hopf invariant, various authors in hydrodynamics introduced a Hopf invariant $H(X,Y) \in \mathbb{R}$ for pairs (X,Y) of divergence-free vector fields on a three-dimensional homology sphere M. Arnold [1] interpreted this Hopf invariant as the average over $M \times M$ of the long-time asymptotic linking numbers of pairs of flow lines for X and Y. In order to make this interpretation precise, one has to find a coherent way of closing long pieces of flow lines to form loops so that one can evaluate the corresponding linking numbers and compute their long-time asymptotics. Arnold's construction of such a system of short path works for suitably generic vector fields, cf. [2], but there are technical problems if one allows vector fields with degenerate zeroes. A slightly different construction covering all divergence-free vector fields was given by the second author in [14].

A variation of Arnold's construction formulated for nonsingular flows on S^3 with invariant measures is contained in [5]. Higher-dimensional generalisations are somewhat elusive, because it is not at all clear how to parameterise and "close" higher-dimensional open submanifolds in

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order to define linking numbers. In this paper we define the average asymptotic linking number of a divergence-free vector field and a measured codimension two foliation. We close up pieces of flow lines of the vector field as in [14], but we do not parameterise or close the leaves of the foliation. Intuitively, parameterisation and "closing" is done by the invariant measure. Thinking of a divergence-free flow as a (singular) one-dimensional foliation with a holonomy-invariant transverse measure, we recover the construction of [1, 14]. A different generalisation of Arnold's construction has been proposed by Rivière [12].

In section 3 we shall consider the linking of the flow lines of a divergence-free vector field with a null-homologous closed oriented submanifold N of codimension two. On three-manifolds, this situation was already considered by Arnold [1], but our approach is different. For manifolds with vanishing first Betti number we prove the existence of suitable systems of short paths which one can use to close the flow lines of the vector field. The resulting average asymptotic linking number is given by a Hopf-type integral. It is then clear how to generalise further and replace the closed submanifold N by a measured foliation $(\mathcal{F}, \nu)$ of codimension two, whose Ruelle–Sullivan cycle is null-homologous. This is carried out in section 4. The discussion there is motivated in part by the work of Arnold, Khesin [8], Novikov and others on higher-dimensional generalisations of Arnold's construction described in [2] (Chapter III, 7.B).

In section 5 we apply our construction to give an interpretation of the four-dimensional Godbillon-Vey invariant discussed in [9] as an average asymptotic linking number of a vector field and a measured codimension two foliation. For this it is important that the constructions of section 4 work for singular foliations.

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2. Preliminaries

Let M^n be a smooth closed oriented n-manifold. Consider smooth closed oriented submanifolds N_1^k and N_2^l of M, with $k + l = n - 1$. If N_1 and N_2 are null-homologous over $\mathbb{R}$, they have a well-defined linking number defined as follows. Let S_1 be a real oriented $(k+1)$-chain with $\partial S_1 = N_1$, and define $\mathrm{lk}(N_1, N_2)$ to be the algebraic intersection number of S_1 and N_2, which by assumption have complementary dimensions. This linking number is always rational. If N_1 and N_2 are null-homologous over $\mathbb{Z}$, then their linking number is integral. In any case, we have $\mathrm{lk}(N_1, N_2) = (-1)^{(k+1)(l+1)}\mathrm{lk}(N_2, N_1)$.

There is a classical expression for these linking numbers as integrals of certain differential forms. If W_i is a tubular neighbourhood of N_i, we can choose a closed form η_i on W_i representing the compactly supported Poincaré dual of N_i in W_i, and then extend this form by zero to all of M. The assumption that $[N_1]$ vanishes in real homology means that the extension of η_1 to M, also denoted η_1, is exact. Let α_1 be a primitive and define

$$H(N_1, N_2) = \int_M \alpha_1 \wedge \eta_2 \ . \tag{1}$$

If we chose a different primitive α_1' for η_1, then

$$\int_M \alpha_1' \wedge \eta_2 - \int_M \alpha_1 \wedge \eta_2 = \int_M (\alpha_1' - \alpha_1) \wedge \eta_2 \ .$$

This vanishes because $\alpha_1' - \alpha$ is closed and η_2 is exact as N_2 is also assumed to be null-homologous. If we make a different choice, η_2', for η_2, then $\eta_2' - \eta_2 = d\beta$, for a β with support in W_2. Then

$$\int_M \alpha_1 \wedge \eta_2' - \int_M \alpha_1 \wedge \eta_2 = \int_M \alpha_1 \wedge d\beta = \pm \int_M d\alpha_1 \wedge \beta = \pm \int_M \eta_1 \wedge \beta \ .$$

Now the right hand side vanishes because η_1 and β have support in W_1 and W_2 respectively, which can be chosen to be disjoint.

Thus we have shown that $H(N_1, N_2)$ is independent of choices, except possibly for the choice of η_1. However, we can interchange the roles of η_1 and η_2 because we have $H(N_1, N_2) = (-1)^{(k+1)(l+1)}H(N_2, N_1)$. Thus $H(N_1, N_2)$ is independent of all choices and only depends on the submanifolds N_1 and N_2, as suggested by the notation. By making a convenient choice for η_2 as in [3], one sees that

$$\int_M \alpha_1 \wedge \eta_2 = \int_{N_2} \alpha_1 \tag{2}$$

and that these integrals equal the linking number $\mathrm{lk}(N_1, N_2)$. An instance of this equality is the differential form interpretation of the Hopf invariant, and we shall refer to either $\int_M \alpha_1 \wedge \eta_2$ or $\int_{N_2} \alpha_1$ as a Hopf-type integral.

The linking number of two disjoint closed oriented submanifolds can be expressed through so-called linking forms (see [11, 14]). A *linking form* L on an n-manifold M is a double form on $M \times M$ with the property that whenever the linking number of two oriented closed submanifolds N_1 and N_2 is well-defined, we have

$$\mathrm{lk}(N_1, N_2) = \int_{N_1}\int_{N_2} L \ .$$

A linking form can be constructed as follows. We choose a Riemannian metric on M, and define H to be the projection operator mapping a differential form to its harmonic part. Then, for every degree i, the Green's operator

$$G\colon \Omega^i(M) \longrightarrow (\mathcal{H}^i)^\perp$$

is defined to map a differential form α of degree i to the unique i-form ω that is perpendicular to all harmonic forms and solves the equation

$$\Delta\omega = \alpha - H(\alpha) \ .$$

In other words, G is characterised by the properties $HG = 0$ and $\Delta G = \mathrm{Id} - H$. It can be written in the form

$$G(\alpha)(x) = \int_{y\in M} \alpha(y) \wedge *_y g(x,y)$$

for a suitable double form $g(x,y)$ on $M\times M$ which is smooth away from the diagonal and has a pole of order $n-2$ along the diagonal, cf. [11] §31. Here $*_y$ denotes the Hodge star operator with respect to the second factor of $M\times M$.

We denote by $(-1)^\epsilon$ the linear operator on double forms acting on decomposable double forms $\omega(x)\cdot\eta(y)$ with $\eta\in\Omega^s(M)$ by multiplication with $(-1)^{(n-s)s}$. In later sections we will be concerned only with the case $s = n-1$. (This is due to the fact that the Poincaré duals of one-dimensional submanifolds of M have degree $n-1$.) We define the double form L on $M\times M$ by

$$L(x,y) = (-1)^\epsilon *_y d_y g(x,y) \ . \tag{3}$$

Proposition 1. *The double form $L(x,y)$ is a linking form. Denoting by r the Riemannian distance function, L has a singularity of order $(r(x,y))^{1-n}$ along the diagonal in $M\times M$ and is smooth elsewhere. It has the following additional property: for every i-form α there exists an $(i-1)$-form h such that*

$$\int_{y\in M} L(x,y)\wedge d\alpha(y) = \alpha(x) - H(\alpha)(x) + dh(x) \ . \tag{4}$$

Proof. By the definition of G, and because G commutes with Δ, we have

$$G(d^*d\alpha) = \alpha - H(\alpha) - G(dd^*\alpha) \ .$$

As G commutes with d we can set $h = -G(d^*\alpha)$ to obtain

$$\int_{y\in M} d\alpha(y) \wedge *_y d_y g(x,y) = G(d^* d\alpha) = \alpha(x) - H(\alpha)(x) + dh(x) \ .$$

If we change the order of the factors in the integrand, we have to multiply by $(-1)^\epsilon$ and we obtain (4).

That L is a linking form can be shown as in the proof of Theorem 3 in [14]. We briefly indicate the argument. Choosing W_i, η_i and α_i as above, and using (4), we have

$$\begin{aligned} lk(N_1,N_2) &= \int_M \alpha_1 \wedge \eta_2 \\ &= \int_{x\in M} \left(H(\alpha_1)(x) - dh(x) + \int_{y\in M} L(x,y)\wedge \eta_1(y) \right) \wedge \eta_2(x) \ . \end{aligned}$$

The integrals

$$\int_M H(\alpha_1)\wedge \eta_2 \quad \text{and} \quad \int_M dh \wedge \eta_2$$

vanish by Stokes's theorem because η_2 is exact. As η_i has support in W_i, we obtain

$$lk(N_1,N_2) = \int_{x\in W_2} \left(\int_{y\in W_1} L(x,y)\wedge \eta_1(y) \right) \wedge \eta_2(x) \ .$$

Now the W_i can be taken to be disjoint and making a good choice for the η_i the above integral reduces to

$$\int_{x\in N_2} \int_{y \in N_1} L(x,y) \ .$$

The claim about the order of the singularity along the diagonal follows from what we said above about the singularity of g. □

We shall also need to use the mean ergodic theorem as in [14].

Theorem 2 ([6]). *Let f be an L^1-function on the compact manifold M. Let ϕ_t be a differentiable flow on M preserving a given volume form μ.*

(1) *The limit*

$$\tilde f(x) = \lim_{t\to\infty} \frac{1}{t} \int_{s=0}^{t} f(\phi_s(x))ds$$

exists in the L^1-sense and is an integrable function.

(2) *The integral of $\tilde f$ satisfies*

$$\int_M \tilde f \mu = \int_M f\mu \ .$$

3. Linking numbers between vector fields and submanifolds

Let M^n be a smooth closed oriented n-manifold, with $n \geq 3$. We fix once and for all a Riemannian metric and the corresponding linking form L as in Proposition 1.

Instead of considering the linking between two closed oriented submanifolds N_1 and N_2, we shall replace N_1 by loops formed by closing up the flow lines of a divergence-free vector field X, and consider how these loops link with an oriented submanifold N^{n-2} playing the role of N_2 above. As before, we assume that $N \subset M$ is null-homologous over $\mathbb{R}$. To ensure that all the loops are null-homologous, we assume that the first Betti number of M vanishes.

Let μ be a volume form on M, and X a vector field that is divergence-free with respect to μ, i. e. such that $L_X\mu = 0$. Then the $(n-1)$-form $\eta = i_X\mu$ is closed, and our assumption that the first Betti number of M vanishes implies, via Poincaré duality, that η must be exact. Let α be a primitive. Then, generalising (2), we can define a Hopf-type integral for X and N by setting

$$H(X,N) = \int_N \alpha \ .$$

If we choose a different primitive α' for $\eta = i_X\mu$, then

$$\int_N \alpha' - \int_N \alpha = \int_N \alpha' - \alpha$$

vanishes because $\alpha' - \alpha$ is closed and N is assumed to be homologous to zero. Thus $H(X,N)$ is well-defined.

We want to interpret this integral as an average of asymptotic linking numbers of flow lines of X with N. To do so, we fix once and for all a "system of short paths" connecting any pair of points $p, q \in M$. The flow of X will be denoted by ϕ_t.

Definition 3. A set Σ of piecewise differentiable paths in M is *a system of short paths* if it has the following properties:

(1) For any two points $p, q \in M$ there is exactly one path $\sigma(p,q) \in \Sigma$ starting at p and ending at q.
(2) The paths depend continuously on their endpoints almost everywhere.
(3) The limit

$$\lim_{t\to\infty} \frac{1}{t} \int_{y\in\sigma(\phi_t(x),x)} \int_{p\in N} L(p,y) = 0 \qquad (5)$$

exists in the L^1-sense.

Theorem 4. *A system of short paths exists. It can be chosen independently of the vector field X.*

Proof. Let $C(y) = \int_{p \in N} L(p,y)$. This is a well-defined L^1-form on M. Since M is compact it can be covered by a finite number of geodesic balls U_j, $j = 1, \dots, r$. In each geodesic ball we fix a basepoint $u_j \in U_j$ such that the following conditions are satisfied.

(1) For every pair k, j there is a path γ_{kj} parametrised by $[0,1]$ joining u_k and u_j such that the integral

$$\int_{y \in \gamma_{kj}} |C(y)| := \int_0^1 \left| i_{\frac{\partial \gamma_{kj}}{\partial s}} C(\gamma_{kj}(s)) \right| ds \tag{6}$$

is finite.

(2) Let $\sigma(x, u_j)$ denote the unique geodesic in U_j between $x \in U_j$ and u_j. For all j the integral

$$\int_{x \in U_j} \int_{y \in \sigma(x, u_j)} |C(y)| \mu(x) \tag{7}$$

is finite.

The second condition is satisfied if all the u_j are outside of N. Since N has codimension two, both conditions hold for a generic choice of the u_j.

For all $x \in M$ fix a number $n(x)$ such that $x \in U_{n(x)}$ and $n(x)$ is locally constant on a dense open subset of M. Let $p, q \in M$. We define a piecewise differentiable path $\sigma(p,q)$ joining p and q as follows. The first segment of $\sigma(p,q)$ is the unique geodesic in $U_{n(p)}$ between p and $u_{n(p)}$, the second segment is $\gamma_{n(p),n(q)}$ and the third is the unique geodesic in $U_{n(q)}$ starting at $u_{n(q)}$ and ending at q. Note that for x with $n(x) = j$, the short path between x and u_j is the unique geodesic in U_j joining x and u_j, this justifies the notation σ for both objects.

We define Σ to be the set of paths obtained this way. For each $p, q \in M$ there is a unique path with starting point p and end point q. By construction, the paths depend in a continuous way on their starting and end points on an open dense subset of $M \times M$.

Dividing the paths into their differentiable pieces we obtain

$$\begin{aligned}\int_{x\in M}\left|\int_{y\in\sigma(\phi_t(x),x)} C(y)\right|\mu(x) \leq & \int_{x\in M}\int_{y\in\sigma(\phi_t(x),u_{n(\phi_t(x))})}|C(y)|\mu(x)\\ &+\int_{x\in M}\int_{y\in\gamma_{n(\phi_t(x)),n(x)}}|C(y)|\mu(x)\\ &+\int_{x\in M}\int_{y\in\sigma(u_{n(x)},x)}|C(y)|\mu(x)\ .\end{aligned}$$

The first and the third summand on the right hand side are in fact equal. To see this, apply the volume-preserving transformation $x\mapsto\phi_{-t}(x)$ to the first summand. In particular these two summands do not depend on t. The second summand is bounded above by

$$\max_{i,j}\left(\int_{y\in\gamma_{ij}}|C(y)|\right)\int_{x\in M}\mu(x)$$

and the third summand is bounded above by

$$\sum_{i=1}^{r}\int_{x\in U_i}\int_{y\in\sigma(u_i,x)}|C(y)|\mu(x)\ .$$

Thus Σ meets the third condition of Definition 3. □

If x is a point in M and $t\in\mathbb{R}$, denote by $\phi(x,t)$ the flow line of X generated by x in the time-interval $[0,t]$. We shall denote by $\gamma(x,t)$ the closed loop obtained by connecting the endpoints x and $\phi_t(x)$ of $\phi(x,t)$ by the path $\sigma(\phi_t(x),x)$.

The following is true for dimension reasons:

Lemma 5. *Let $t\in\mathbb{R}$ be fixed. Then for μ-almost all $x\in M$, the piecewise differentiable curve $\gamma(x,t)$ is embedded in $M\setminus N$.*

Given this Lemma and our assumptions that the first Betti number of M vanishes and that N is null-homologous over $\mathbb{R}$, we can define the linking number $\mathrm{lk}(\gamma(x,t),N)$ for almost all x and t. We then have:

Proposition 6. *The limit*

$$\mathrm{lk}(x,N)=\lim_{t\to\infty}\frac{1}{t}\mathrm{lk}(\gamma(x,t),N)$$

exists in the L^1-sense. It is an integrable function on M which does not depend on the chosen system of short paths.

Proof. By Proposition 1 we have

$$\lim_{t\to\infty}\frac{1}{t}\mathrm{lk}(\gamma(x,t),N)=\lim_{t\to\infty}\frac{1}{t}\int_{y\in\gamma(x,t)}\int_{p\in N}L(p,y)\ .$$

Using the third property of the system of short paths, we find that the right hand side equals

$$\lim_{t\to\infty} \frac{1}{t} \int_{s=0}^{t} \int_{p\in N} i_X L(p, \phi_s x) ds \ .$$

Since $L(p, \cdot)$ is an integrable form on M for every $p \in M$, we can apply the mean ergodic theorem, Theorem 2. Hence the limit exists. It is clearly independent of the system of short paths. □

Using this, we can finally define the average asymptotic linking number of the vector field X with the submanifold N, by setting

$$\mathrm{lk}(X, N) = \int_M \mathrm{lk}(x, N)\mu \ .$$

The analog of the theorem proved in [1, 14] for vector fields on 3-manifolds is:

Theorem 7. *Let M be a closed oriented n-manifold with $b_1(M) = 0$. Let X be a divergence-free vector field on M, and $N \subset M$ a closed oriented submanifold of codimension 2 which is null-homologous over $\mathbb{R}$. Then the average asymptotic linking number of the orbits of X with N exists and equals a Hopf-type integral:*

$$\mathrm{lk}(X, N) = H(X, N) \ .$$

Proof. We have seen in the proof of Proposition 6 that

$$\mathrm{lk}(x, N) = \lim_{t\to\infty} \frac{1}{t} \int_{s=0}^{t} \int_{p\in N} i_X L(p, \phi_s x) ds \ .$$

By the second part of the mean ergodic theorem, Theorem 2, we find

$$\begin{aligned}
\mathrm{lk}(X, N) &= \int_{x\in M} \left(\lim_{t\to\infty} \frac{1}{t} \int_{s=0}^{t} \int_{p\in N} i_X L(p, \phi_s x) ds \right) \mu(x) \\
&= \int_{x\in M} \int_{p\in N} i_{X(x)} L(p, x) \wedge \mu(x) \\
&= \int_{x\in M} \int_{p\in N} L(p, x) \wedge i_{X(x)}\mu(x) \ ,
\end{aligned}$$

where $i_X\mu = d\alpha$. We now apply Proposition 1 to the last integral. The integral of the harmonic term in Proposition 1 over N is zero because N is null-homologous, and by Stokes's theorem the exact term plays no role. Hence we obtain the desired result

$$\mathrm{lk}(X, N) = \int_N \alpha = H(X, N) \ .$$

□

To end this section we consider some geometric examples.

Example 8. Let (M,ω) be a symplectic manifold of dimension $2n$. Then ω^n is a volume form on M. Consider a smooth function H on M and the corresponding Hamiltonian vector field X_H. This is the unique vector field on M satisfying $dH = i_{X_H}\omega$. Then X_H is divergence-free with respect to ω^n since

$$L_{X_H}\omega^n = di_{X_H}\omega^n = d(ndH \wedge \omega^{n-1}) = 0 \ .$$

Moreover, the $(2n-2)$–form $nH\omega^{n-1}$ satisfies

$$d(nH\omega^{n-1}) = ndH \wedge \omega^{n-1} = n(i_{X_H}\omega) \wedge \omega^{n-1} = i_{X_H}(\omega^n) \ .$$

Let N be a null-homologous submanifold of codimension 2 in M. If $b_1(M) = 0$, the linking number of X with N is given by

$$\mathrm{lk}(X,N) = n\int_N H\omega^{n-1} \ . \tag{8}$$

For a fixed Hamiltonian vector field the function generating it is well-defined only up to the addition of locally constant functions. By Stokes's theorem this ambiguity in the choice of H does not change the value of the integral in (8) since N is homologous to zero.

Example 9. Consider a manifold of dimension $2n+1$ with a contact form $\alpha \in \Omega^1(M)$. This means that $\alpha \wedge (d\alpha)^n$ is a volume form on M. There is a unique vector field X, called the Reeb vector field, with the properties $\alpha(X) = 1$ and $i_X d\alpha = 0$. Because

$$L_X(\alpha \wedge (d\alpha)^n) = di_X(\alpha \wedge (d\alpha)^n) = d(d\alpha)^n = 0 \ ,$$

the Reeb vector field is divergence–free with respect to the volume form $\alpha\wedge(d\alpha)^n$. The $(2n-1)$–form $\alpha\wedge(d\alpha)^{n-1}$ is a primitive of $i_X(\alpha\wedge(d\alpha)^n) = (d\alpha)^n$. Thus, if $b_1(M) = 0$, the linking number of the Reeb vector field X with a null-homologous submanifold N of codimension two is

$$\mathrm{lk}(X,N) = \int_N \alpha \wedge (d\alpha)^{n-1} \ .$$

In particular it is nonzero if N is a contact submanifold.

Example 10. Consider $M = S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2$ and the map

$$\begin{aligned} f\colon S^3 \times S^3 &\longrightarrow \mathbb{C} \\ (z,w) &\longmapsto \langle z,w\rangle = z_0\bar{w}_0 + z_1\bar{w}_1 \ . \end{aligned}$$

As 0 is a regular value of f the preimage $N = f^{-1}(0)$ is a smooth submanifold of M of dimension four. The map

$$\begin{aligned} S^3 \times S^1 &\longrightarrow N \\ ((z_0,z_1),\lambda) &\longmapsto ((z_0,z_1),(\lambda\bar{z}_1, -\lambda\bar{z}_0)) \end{aligned}$$

is a diffeomorphism. Since $H_4(M) = 0$, N is null-homologous. We now construct a submanifold in M whose boundary is N. Consider the set $[0, \infty) \subset \mathbb{C}$ of nonnegative real numbers. Let (z, w) be a point in M that is mapped by f to a positive real number c. Then

$$\left.\frac{d}{dt}\right|_{t=0} f(e^{it}z, w) = c\partial_y \ .$$

Together with the fact that 0 is a regular value of f, this shows that f is transversal to $[0, \infty)$. Thus $f^{-1}([0, \infty))$ is a five-dimensional submanifold of M whose boundary is N.

Denote by pr_1 the projection of M onto the first factor and by pr_2 the projection onto the second factor. We define the vector field H_1 on M as the unique vector field with the property that $\mathrm{pr}_{1*}(H_1)$ is the Hopf vector field on S^3 and $\mathrm{pr}_{2*}(H_1) = 0$. The vector field H_2 is defined similarly with the roles of pr_1 and pr_2 interchanged. For arbitrary constants a and b the vector field $aH_1 + bH_2$ preserves the canonical volume form on $M = S^3 \times S^3$. The flow of this vector field at time t maps (z, w) to $(e^{iat}z, e^{ibt}w)$. If the difference $a - b$ is a rational multiple of 2π, then all flow lines are closed, otherwise all flow lines are open.

We consider now the flow line with starting point (z, w). At time t we find

$$\langle z(t), w(t)\rangle = e^{i(a-b)t}\langle z, w\rangle \ .$$

If $a \neq b$ this means that every flow line emerging from a point of $M \setminus N$ intersects $f^{-1}([0, \infty))$ transversely during the time interval $[0, \frac{2\pi}{|a-b|}]$. Then an intersection of the flow line with $f^{-1}([0, \infty))$ occurs periodically after time intervals of length $\frac{2\pi}{|a-b|}$. The intersection is positive if $a > b$ and negative if $a < b$. Thus we have shown

$$\mathrm{lk}((z, w), N) = \frac{a - b}{2\pi}$$

if $(z, w) \notin N$.

For flow lines with starting point in N the linking number with N is not well-defined.

For the asymptotic linking number of $aH_1 + bH_2$ with N we find

$$\mathrm{lk}(aH_1 + bH_2, N) = \frac{a - b}{2\pi}(\mathrm{vol}(S^3))^2 = 2(a - b)\pi^3 \ .$$

4. Linking numbers between vector fields and measured foliations

As before, we assume that M is a closed oriented manifold with $b_1(M) = 0$, μ is a volume form on M, and X a vector field that is divergence-free with respect to μ, i. e. such that $L_X\mu = 0$. Then the $(n-1)$-form $\eta = i_X\mu$ is closed, and our assumptions imply that η must be exact. Let α be a primitive.

We generalise the discussion in section 3 by replacing the submanifold N by an oriented codimension 2 foliation $\mathcal{F}$ with a holonomy-invariant transverse measure ν. This defines a current

$$\begin{aligned} C(\mathcal{F},\nu)\colon \Omega^{n-2}(M) &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_M \omega\wedge\nu \ , \end{aligned}$$

where

$$\int_M \omega\wedge\nu = \int_T (\int_{\mathcal{F}} \omega) d\nu$$

is defined by decomposing ω using a partition of unity subordinate to a finite atlas of foliation charts for $(M,\mathcal{F})$, integrating the summands over the plaques of $\mathcal{F}$ in the charts, and then integrating the result over the transversals T using ν. The double integral is independent of the choices of charts and partition of unity because ν was assumed to be holonomy-invariant. See [13, 4] for the details of this construction.

The current $C(\mathcal{F},\nu)$ is closed, and is called the Ruelle–Sullivan cycle of the invariant measure ν. It will play the role of the submanifold N in section 3. The assumption that N be null-homologous over $\mathbb{R}$ is then replaced by the assumption that the Ruelle–Sullivan cycle is null-homologous: $[C(\mathcal{F},\nu)] = 0 \in H_{n-2}(M,\mathbb{R})$.

The Ruelle–Sullivan cycle $C(\mathcal{F},\nu)$ is continuous with respect to the C^0–topology on continuous forms. In general it is not possible to extend its domain of definition to L^1–forms. Nevertheless, it is possible to define $C(\mathcal{F},\nu)$ on integrable double forms $F(x,y)$ with the property that

$$\int_{y\in M} |F(x,y)|_y \mu(y)$$

is a continuous form in the variable $x \in M$. Here we expand F as a double form in the variables x and y, and take the x-component multiplied by the norm of the y-component with respect to our fixed Riemannian metric. Thus $|F(x,y)|_y$ is a differential form which is a product of dx_i, but whose coefficient function also depends on y. As we integrate along the leaves of $\mathcal{F}$, we are only concerned with those summands of $F(x,y)$ whose degree in the variable x equals the rank of the foliation, and we require $|F(x,y)|_y$ to be either zero or to induce the given orientation when restricted to the leaf of $\mathcal{F}$ through x. By Proposition 1 the linking form L satisfies this integrability condition. The form $C(\mathcal{F},\nu)(F(x,y))$ is obtained by performing the integrations in the Ruelle–Sullivan cycle with respect to the first variable. The result is an integrable form as can be shown with Fubini's theorem

$$\int_{y\in M} \left|C(\mathcal{F},\nu(x))\big(F(x,y)\big)\right| \mu(y) \leq C(\mathcal{F},\nu(x)) \left(\int_{y\in M} |F(x,y)|_y \, \mu(y)\right) .$$

We define a Hopf-type integral for X and $(\mathcal{F},\mu)$ by setting

$$H(X,\mathcal{F},\nu) = \int_M \alpha\wedge\nu = C(\mathcal{F},\nu)(\alpha) .$$

This is independent of the choice made for α because the Ruelle–Sullivan cycle is assumed to be null-homologous.

We want to interpret this integral as an average of asymptotic linking numbers of flow lines of X with $\mathcal{F}$. To do so, we need again a suitable system of short paths. We use the same notation for the paths and closed-up flow lines as before.

Definition 11. A *system of short paths* in M is a set Σ of piecewise differentiable paths with the following properties:

(1) For every pair of points $p,q\in M$ there is exactly one oriented path $\sigma(p,q)\in\Sigma$ having starting point p and end point q.
(2) The paths depend continuously on their starting and end points almost everywhere.
(3) The limit

$$\lim_{t\to\infty}\frac{1}{t}\int_{y\in\sigma(\phi_t(x),x)} C(\mathcal{F},\nu(p))(L(p,y)) = 0 \tag{9}$$

exists in the L^1–sense.

In the case when the holonomy-invariant measure ν is given by a smooth differential form β, the Ruelle–Sullivan cycle is given by

$$C(\mathcal{F},\nu)(\omega) = \int_M \omega\wedge\beta .$$

The proof of Theorem 4 in [14] generalises verbatim to this situation and gives:

Theorem 12. *If the transverse measure is given by a smooth holonomy-invariant 2–form β, then a set of length-minimizing geodesics is a system of short paths.*

More generally, we have:

Theorem 13. *Let $\mathcal{F}$ be an oriented foliation with an arbitrary holonomy-invariant transverse measure ν. Then there exists a system of short paths.*

Proof. We want to generalise the proof of Theorem 4. We use the notation introduced there, except that we now define $C(y) = C(\mathcal{F}, \nu(p))\big(L(p,y)\big)$. If we can satisfy the two conditions for the choice of the base points $u_i \in U_i$ (with the generalised definition of $C(y)$) in the proof of Theorem 4, then we can construct a system of short paths Σ just as in the proof of Theorem 4.

Fix arbitrary points $\tilde{u}_i \in U_i$ for all $1 \leq i \leq r$ and paths $\tilde{\gamma}_{ij}$ without self intersection joining $\tilde{u}_i$ and $\tilde{u}_j$. For each pair $i \neq j$, extend the velocity vector field along $\tilde{\gamma}_{ij}$ to a vector field on M whose time-one-flow transports a small ball contained in U_i around $\tilde{u}_i$ to another ball contained in U_j around $\tilde{u}_j$. Because $C(y)$ is an integrable form on M, for μ–almost every starting point in a ball around $\tilde{u}_i$ the integral in (6) exists. Hence for almost every choice of u_i in the ball around $\tilde{u}_i$ we meet the first condition. The γ_{ij} are the time-one-flowlines.

Also the second condition for the u_i is satisfied outside of a set of measure zero. To see this, recall that the Ruelle–Sullivan cycle can be represented as a finite sum such that every summand is an integral with respect to a product measure. Apply Fubini's theorem and the triangle inequality to the integral

$$\begin{aligned} (10)\quad & \int_{u\in U_i}\left(\int_{\{x|n(x)=i\}}\int_{y\in\sigma(x,u)} \left|C(\mathcal{F},\nu(p))\big(L(p,y)\big)\right| \ \mu(x)\right)\mu(u) \\ & \leq C(\mathcal{F},\nu(p))\left(\int_{s=0}^{1}\int_{u\in U_i}\int_{\{x|n(x)=i\}}\left|i_{\frac{\partial\sigma}{\partial s}}L\big(p,\sigma(x,u)(s)\big)\right|\mu(x)\ \mu(u)ds\right) . \end{aligned}$$

If the last expression is well-defined, then by Fubini's theorem the expression obtained by dropping the integration with respect to u is well-defined for μ–almost every choice of $u \in U_i$. Thus we have to show that the form we apply the Ruelle–Sullivan cycle to is continuous. Consider

$$(11)\qquad \int_{y\in U_i}\int_{\{x|n(x)=i\}}\left|i_{\frac{\partial\sigma}{\partial s}}L\big(p,\sigma(x,y)(s)\big)\right| \ \mu(x)\ \mu(y)$$

with $p \in M$. For fixed $s \in [0,1]$ the integrand has at most a pole of order $n-1$ along an n-dimensional submanifold (the solutions of $\sigma(x,y)(s) = p$) in the $2n$-dimensional product manifold $U_i \times \{x|n(x) = i\}$. This shows that the integral (11) is well-defined. Furthermore, the integral depends continuously on s and p. In particular, it does exist for the boundary values $s = 0$ and $s = 1$. This shows that we apply the Ruelle–Sullivan cycle to a continuous form and thereby justifies the application of the Ruelle–Sullivan cycle in (10).

So, if we choose $(u_1, \ldots, u_r) \in U_1 \times \ldots \times U_r$ outside of a set of measure zero, we obtain a system of short paths as in the proof of Theorem 4. $\square$

Remark 14. So far we have only considered non–singular foliations. In the next section, we will also want to use singular foliations. We therefore point out that, as in [14], Theorem 12 applies equally well to singular foliations with a holonomy-invariant smooth differential form.

Theorem 13 can sometimes be applied to singular foliations. For example, this can be done if the support of the transverse measure has a neighbourhood to which the foliation extends in a nonsingular way. In the situation of the previous section, a single null-homologous submanifold $N \subset M$ can always be extended to a smooth foliation of a whole neighbourhood of N, and we can take the measure given by N, with support N. Then the above theorem can be used instead of Theorem 4.

Now the linking number of $\gamma(x,t)$ with $\mathcal{F}$ is defined by generalising (2) as follows:

Definition 15. The linking number $\mathrm{lk}(\gamma, \mathcal{F}, \nu)$ of a (null-homologous) closed loop γ in M with the measured foliation $(\mathcal{F}, \nu)$ is the evaluation of the Ruelle–Sullivan cycle $C(\mathcal{F}, \nu)$ on a $(n-2)$-form α with the property that $d\alpha$ is Poincaré dual to γ.

As the Ruelle–Sullivan cycle is assumed to be null-homologous, this evaluation is independent of the choice of α.

The following is the adaption of Proposition 6 to this situation:

Proposition 16. *Let $\alpha(x,t)$ be $(n-2)$-forms with the property that $d\alpha(x,t)$ are Poincaré duals for $\gamma(x,t)$. Then the limit*

$$\mathrm{lk}(x, \mathcal{F}, \nu) = \lim_{t\to\infty} \frac{1}{t} \mathrm{lk}(\gamma(x,t), \mathcal{F}, \nu) = \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F}, \nu)(\alpha(x,t))$$

exists in the L^1-sense. It is an integrable function on M which does not depend on the chosen system of short paths.

Proof. By the definition of $\mathrm{lk}(x, \mathcal{F}, \nu)$ and Proposition 1 we find

$$\begin{aligned}\mathrm{lk}(x, \mathcal{F}, \nu) &= \lim_{t\to\infty} \frac{1}{t} \mathrm{lk}(\gamma(x,t), \mathcal{F}, \nu) = \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F}, \nu)(\alpha(x,t)) \\ &= \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F}, \nu(p)) \left(\int_{y\in M} L(p,y) \wedge d\alpha(x,t)(y) \right) .\end{aligned}$$

The harmonic and exact terms in (4) do not contribute because the Ruelle–Sullivan cycle is assumed to be null-homologous. If p does not lie on $\gamma(x,t)$, and hence for almost every $x \in M$, we have

$$\begin{aligned}
\mathrm{lk}(x,\mathcal{F},\nu) &= \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F},\nu(p)) \left(\int_{y\in\gamma(x,t)} L(p,y) \right) \\
&= \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F},\nu(p)) \left(\int_{y\in\phi(x,t)} L(p,y) \right) \\
&= \lim_{t\to\infty} \frac{1}{t} C(\mathcal{F},\nu(p)) \left(\int_0^t i_X L(p,\phi_s(x)) ds \right) .
\end{aligned}$$

The second equality is true because of the definition of the system of short paths. The flow of X preserves the volume form μ and we can apply the mean ergodic theorem. Hence, the limit on the right hand side of

$$\mathrm{lk}(x,\mathcal{F},\nu) = \lim_{t\to\infty} \frac{1}{t} \int_0^t C(\mathcal{F},\nu(p)) \big(i_X L(p,\phi_s(x)) \big) ds$$

exists in the L^1–sense and represents an integrable function on M. It does not depend on the system of short paths. □

Using this, we can finally define the average asymptotic linking number of the vector field X with the measured foliation $(\mathcal{F},\nu)$ by setting

$$\mathrm{lk}(X,\mathcal{F},\nu) = \int_M \mathrm{lk}(x,\mathcal{F},\nu)\mu \ .$$

Theorem 7 generalises as follows:

Theorem 17. *Let M be a closed oriented n-manifold with $b_1(M) = 0$. Let X be a divergence-free vector field on M, and $\mathcal{F}$ an oriented codimension 2 foliation with a transverse measure ν whose Ruelle–Sullivan cycle is null-homologous. Then the average asymptotic linking number of the orbits of X with $(\mathcal{F},\nu)$ exists and equals a Hopf-type integral:*

$$\mathrm{lk}(X,\mathcal{F},\nu) = H(X,\mathcal{F},\nu) \ .$$

Proof. We use the calculations in the proof of Proposition 16 and the mean ergodic theorem. Since $b_1(M) = 0$, $b_{n-1}(M)$ also vanishes. The $(n-1)$–form $i_X\mu$ is closed and hence exact. Choose an $(n-2)$–form α_X such that $d\alpha_X = i_X\mu$. By the mean ergodic theorem and Proposition 16

$$\begin{aligned}
\mathrm{lk}(X,\mathcal{F},\nu) &= \int_{x\in M} \mathrm{lk}(x,\mathcal{F},\nu)\mu(x) \\
&= \int_{x\in M} C(\mathcal{F},\nu(p))(i_X L(p,x))\mu(x) \ .
\end{aligned}$$

Locally, the Ruelle–Sullivan cycle is given by a product measure, hence we can apply Fubini's theorem. We find

$$\begin{aligned}\mathrm{lk}(X,\mathcal{F},\nu) =& C(\mathcal{F},\nu(p))\left(\int_{x\in M} i_X L(p,x)\wedge\mu(x)\right)\\ =& C(\mathcal{F},\nu(p))\left(\int_{x\in M} L(p,x)\wedge i_X\mu(x)\right)\\ =& C(\mathcal{F},\nu(p))(\alpha_X) = H(X,\mathcal{F},\nu)\ ,\end{aligned}$$

where the penultimate equality is due to Proposition 1 and Stokes's theorem. □

Remark 18. The discussion in this section reduces to that of the previous section in the case that the invariant measure ν is given by a closed leaf N. The rest of the foliation $\mathcal{F}$ then plays no role.

Remark 19. For a smooth foliation with a holonomy-invariant measure given by a smooth exact 2-form, one can prove Theorem 17 using Arnold's definition of a system of short paths and the Birkhoff ergodic theorem as in [1], rather than the mean ergodic theorem as above. This was done by Khesin in [8].

Here are two examples for linking numbers between measured foliations and divergence-free vector fields.

Example 20. Let $\mathcal{F}$ be the Reeb foliation on S^3. It has exactly one closed leaf, which is diffeomorphic to T^2. All other leaves are diffeomorphic to $\mathbb{R}^2$ and have linear growth. By a construction going back to Plante, cf. [4, 13], any leaf $\mathcal{L}$ of subexponential growth defines an invariant measure $\mu_{\mathcal{L}}$ with support contained on a union of minimal sets in $\overline{\mathcal{L}}$. For the Reeb foliation one can show easily that, up to a factor of 2π, the measure $\nu_{\mathcal{L}}$ defined by an open leaf equals that defined by the unique closed leaf T^2.

Consider now the product of two Reeb foliations on $S^3\times S^3$. This foliation $\mathcal{F}\times\mathcal{F}$ has codimension two and contains exactly one closed leaf diffeomorphic to T^4. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are open leaves of $\mathcal{F}$, then the leaf $\mathcal{L}_1\times\mathcal{L}_2$ in the product foliation defines the holonomy-invariant transverse measure $\nu_{\mathcal{L}_1}\times\nu_{\mathcal{L}_2}$ for $\mathcal{F}\times\mathcal{F}$. By the discussion above we have the equality

$$\nu_{\mathcal{L}_1}\times\nu_{\mathcal{L}_2} = \frac{1}{4\pi^2}\nu_{T^4}\ .$$

Thus the linking number of a divergence-free vector field with $(\mathcal{F}\times\mathcal{F},\nu_{\mathcal{L}_1}\times\nu_{\mathcal{L}_2})$ is, up to a factor of $4\pi^2$, the same as that with the submanifold $T^4\subset S^3\times S^3$ given by the closed leaf.

Example 21. Next we consider a singular foliation on $\mathbb{C}P^2$. In order to obtain a null-homologous Ruelle–Sullivan cycle we will use signed measures, rather than measures. Consider the affine embedding

$$\begin{aligned} \mathbb{C}^2 &\hookrightarrow \mathbb{C}P^2 \\ (z_0, z_1) &\mapsto [1 : z_0 : z_1] \ . \end{aligned}$$

We foliate $\mathbb{C}^2$ by the product foliation $\mathbb{C} \times \mathbb{C}$ and consider the leaves $\mathcal{L}_1 = \{1\} \times \mathbb{C}$ and $\mathcal{L}_{-1} = \{-1\} \times \mathbb{C}$. Although these leaves are not closed, the counting measure is a well-defined holonomy-invariant transverse measure since the closure of $\mathcal{L}_1$, respectively $\mathcal{L}_{-1}$, is the union of the leaf with $\{[0 : 0 : 1]\}$. From now on we equip $\mathcal{L}_1$ with the counting measure and $\mathcal{L}_{-1}$ with the negative counting measure, i. e. the signed measure which counts each intersection point with -1. We denote this signed measure by $\delta_1 - \delta_{-1}$.

The corresponding Ruelle–Sullivan cycle is

$$\begin{aligned} C(\mathcal{F}, \delta_1 - \delta_{-1}) : \Omega^2(M) &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_{\mathcal{L}_1} \omega - \int_{\mathcal{L}_{-1}} \omega \ . \end{aligned}$$

Here $C(\mathcal{F}, \delta_1 - \delta_{-1})$ is null-homologous by construction.

Consider $N = \{(z, w) \in \mathbb{C}^2 \subset \mathbb{C}P^2 | z \in [-1, 1] \subset \mathbb{R}\}$. The boundary of this submanifold of $\mathbb{C}^2$ is exactly the union of the two leaves $\mathcal{L}_1$ and $\mathcal{L}_{-1}$, the latter with the reversed orientation. The corresponding current

$$\begin{aligned} \Omega^3(\mathbb{C}P^2) &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_N \omega \end{aligned}$$

has the boundary $C(\mathcal{F}, \delta_1 - \delta_{-1})$.

Let μ be a volume form on $\mathbb{C}P^2$ and X a divergence–free vector field. Let α be a primitive of $i_X \mu$. Then the linking number of X with $(\mathcal{F}, \delta_1 - \delta_{-1})$ is

$$\begin{aligned} \mathrm{lk}(X, \mathcal{F}, \delta_1 - \delta_{-1}) &= \int_{\mathcal{L}_1} \alpha - \int_{\mathcal{L}_{-1}} \alpha \\ &= \int_N i_X \mu \ . \end{aligned}$$

This is exactly the flux of X through N.

5. Godbillon-Vey invariants as linking numbers

Recall that the tangent distribution of a smooth codimension 1 foliation $\mathcal{F}$ with trivial normal bundle on a manifold M is the kernel of a non-vanishing 1-form α which, by the Frobenius theorem, satisfies

$$d\alpha = \alpha \wedge \beta \tag{12}$$

for some 1-form β. The 3-form $\beta \wedge d\beta$ is closed, and its cohomology class $GV(\mathcal{F}) \in H^3(M, \mathbb{R})$ is independent of the choices made for α and β. This is the Godbillon-Vey invariant [7] of $\mathcal{F}$.

In the case that M is closed, oriented and 3-dimensional, $GV(\mathcal{F})$ is equivalent to the Hopf integral

$$\int_M \beta \wedge d\beta \ . \tag{13}$$

If we choose an arbitrary volume form μ on M and define a vector field X in M by the formula $i_X\mu = d\beta$, then X is divergence-free with respect to μ, and (13) is just the Hopf invariant $H(X, X)$. If M is a $\mathbb{R}$-homology sphere, then by the "helicity theorem" due to Arnold [1] and the second author [14], this Hopf invariant can be interpreted as the average asymptotic self-linking number of the orbits of X.

Assume now that we have a smooth 1-parameter family of smooth codimension 1 foliations $\mathcal{F}_t$ with trivial normal bundles. Then (12) still holds, but now α and β are functions of the parameter $t \in \mathbb{R}$. We denote the time derivatives by a dot. It was shown by the first author [9] that for every t the 4-form $(\dot\beta \wedge \beta \wedge d\beta)(t)$ is closed, and that its cohomology class $TGV(\mathcal{F}_t) \in H^4(M, \mathbb{R})$ is a well-defined invariant of the family $\mathcal{F}_t$ that is independent of choices.

In the case that M is closed, oriented and 4-dimensional, $TGV(\mathcal{F}_t)$ is equivalent to the integral

$$\int_M \dot\beta \wedge \beta \wedge d\beta(t) \ . \tag{14}$$

We want to give an interpretation of this as an average asymptotic linking number of a suitable vector field and a measured codimension 2 foliation.

Choose an arbitrary volume form μ on M and define a time-dependent vector field X by the formula $i_X\mu = d(\dot\beta \wedge \beta)$.

Differentiating (12), we see that $d\beta \wedge d\beta = 0$ because $d\beta$ is decomposable. Thus, on the open set in M where $d\beta$ does not vanish, its kernel is a 2-dimensional distribution which is integrable because the defining form is closed. We denote by $\mathcal{G}$ the singular codimension 2 foliation defined by $d\beta$. Note that the exact form $d\beta$ defines an invariant transversal measure for $\mathcal{G}$ whose Ruelle–Sullivan cycle is null-homologous. Now we recognise (14) as the Hopf-type integral associated to the vector field X (at time t) and the measured foliation $(\mathcal{G}, d\beta)$ (at the same time t).

To interpret this as an average asymptotic linking number according to Theorem 17, we need to assume $b_1(M) = 0$. As M is closed, oriented and 4-dimensional, it follows that the Euler characteristic of M is positive, and so there cannot be any non-singular codimension 1 foliation $\mathcal{F}$ on M. However, as we have ended up with only a singular foliation for $\mathcal{G}$, there is no harm in allowing $\mathcal{F}$ to be singular as well. So we just assume that $\mathcal{F}_t$ is the kernel of a time-dependent 1-form α satisfying (12), but allow α to have zeroes. Under certain technical assumptions on α, cf. the appendix, the definition of $TGV(\mathcal{F}_t)$ goes through as in the non-singular case, and if $b_1(M) = 0$, then according to Theorem 17 the integral of $TGV(\mathcal{F}_t)$ over M is the average asymptotic linking number $\mathrm{lk}(X, \mathcal{G}, d\beta)$, with X and $\mathcal{G}$ defined as above. We can apply Theorem 17 to the singular foliation $\mathcal{G}$ because the holonomy-invariant measure is given by a smooth form, see Remark 14.

If $\mathcal{F}_t$ is a 1-parameter family of codimension q foliations on M defined by a decomposable q-from α, then setting $d\alpha = \alpha \wedge \beta$ as above, we can consider $\dot{\beta} \wedge \beta \wedge (d\beta)^q$. It was proved in [9] that this is closed, and that its cohomology class $TGV(\mathcal{F}_t) \in H^{2q+2}(M, \mathbb{R})$ is a well-defined invariant of the family $\mathcal{F}_t$. One might be tempted to think that when M is closed oriented of dimension $2q + 2$, then this invariant should always be an average asymptotic linking number, as was proved above for the case $q = 1$. However, this does not seem to be the case. We always have $(d\beta)^{q+1} = 0$, showing that $d\beta$ has at least a 2-dimensional kernel. If $TGV(\mathcal{F}_t)$ doesn't vanish, then $(d\beta)^q \neq 0$ on an open set in M, so that the kernel of $d\beta$ is exactly of rank 2 (on this open set). This means that whenever $q > 1$, the codimension of the cokernel of $d\beta$ is strictly larger than 2, and so there can be no linking number with 1-dimensional flow lines.

Appendix: Singular foliations and forms with the division property

In the last section we have had to consider codimension one foliations defined by one-forms α which have zeroes. In this appendix we explain how the definitions of the Godbillon-Vey invariant GV and of the invariant TGV of families extend to this situation. As in the rest of this paper, all forms are smooth of class C^∞.

Following Moussu [10], we consider the following:

Definition 22. A one-form α has the division property if $\omega \wedge \alpha = 0$ for a smooth $\omega \in \Omega^k(M)$ with $0 < k < \dim(M)$ implies $\omega = \alpha \wedge \beta$ for some smooth $\beta \in \Omega^{k-1}(M)$.

Note that the division property then implies that β is unique up to the addition of multiples of α.

Nowhere vanishing 1-forms have the division property, as do some classes of singular 1-forms. For example, it is easy to see that if near every point where it vanishes, α is locally the differential of a Morse function, then it has the division property. More generally, Moussu [10] proved that 1-forms with only algebraically isolated zeros have the division property. Algebraic isolation of the zeros means that in local coordinates at a zero the coefficient functions of the one-form span an ideal of finite codimension in the algebra of germs of functions.

The calculation showing the well-definedness of the Godbillon-Vey invariant [7] of a codimension one foliation does not require the defining form α to be non-zero, but rather requires only that it have the division property. Thus we have:

Theorem 23. *Let α be a smooth 1-form with $\alpha \wedge d\alpha = 0$, and $\mathcal{F}$ its kernel foliation. If α has the division property then there is a 1-form β with*

$$d\alpha = \alpha \wedge \beta \ . \tag{15}$$

The 3-form $\beta \wedge d\beta$ is closed and its cohomology class $GV(\mathcal{F}) \in H^3(M, \mathbb{R})$ is independent of the choices made for α and β, as long as α is changed only by multiplication with a nowhere vanishing function.

This generalises to families in the following way:

Theorem 24. *Let α be a smooth 1-form on M with $\alpha \wedge d\alpha = 0$ and which depends smoothly on a parameter $t \in \mathbb{R}$. Let $\mathcal{F}_t$ be the family of kernel foliations. If for every t the form α has the division property, then the 4-forms $(\dot\beta \wedge \beta \wedge d\beta)(t)$ are closed, and their cohomology classes $TGV(\mathcal{F}_t) \in H^4(M, \mathbb{R})$ are independent of the choices made for α and β, as long as α is changed only by multiplication with a nowhere vanishing function.*

Proof. Once we have the forms β and $\dot\beta$, checking the claims in the theorem is done exactly as in the nonsingular case, compare [9].

In the case when α has no zeros, it is easy to see that if α depends smoothly on t, then one can also choose β in (15) to depend smoothly on t. In the singular case, the smooth dependence of β on t is less obvious. However, if $\alpha(0)$ has only algebraically isolated zeros, then the same is true for $\alpha(t)$ for all t sufficiently close to 0, and in this case β can be chosen to depend smoothly on t by adapting Moussu's argument [10].

One can also give an alternative treatment for all divisible forms as follows. Recall from [9] the identity

$$\dot{\beta} \wedge \beta \wedge d\beta = d(\dot{\alpha} \wedge \beta \wedge \gamma) - \dot{\alpha} \wedge \beta \wedge \alpha \wedge \delta \ , \tag{16}$$

where γ and δ are 1-forms constructed by successive exterior differentiation of (15) and the division property of α. This shows that $TGV(\mathcal{F}_t)$ can be defined using only the smooth dependence of α on t, and well-definedness follows as in [9] using the division property. $\square$

References

1. V. I. Arnold, *The asymptotic Hopf invariant and its applications*, Sel. Math. Sov. **5** (1986), 327–345.
2. V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Springer Verlag 1998.
3. R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer Verlag 1982.
4. A. Candel and L. Conlon, *Foliations I*, Amer. Math. Soc. Providence, R.I. 2000.
5. G. Contreras and R. Iturriaga, *Average linking numbers*, Ergod. Th. & Dynam. Sys. **19** (1999), 1425–1435.
6. N. Dunford and J. T. Schwartz, *Linear operators I*, Interscience Publishers 1958.
7. C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension* 1, C. R. Acad. Sci. Paris **273** (1971), 92–95.
8. B. A. Khesin, *Ergodic interpretation of integral hydrodynamic invariants*, J. Geom. and Physics **9** (1992), 101–110.
9. D. Kotschick, *Godbillon-Vey invariants for families of foliations*, Preprint arXiv:math.GT/0111137.
10. R. Moussu, *Sur l'existence d'intégrales premières pour un germe de forme de Pfaff*, Ann. Inst. Fourier **26** (1976), 171–220.
11. G. de Rham, *Differentiable manifolds*, Springer Verlag 1984.
12. T. Rivière, *High-dimensional helicities and rigidity of linked foliations*, Preprint.
13. D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. math. **36** (1976), 225–255.
14. T. Vogel, *On the asymptotic linking number*, Proc. Amer. Math. Soc. (to appear).

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