THE MANIFOLD STRUCTURE OF G/H

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Let G be a connected Lie group, and let H be a closed subgroup of G. We show that the coset space G/H has the structure of a C^∞ manifold of dimension dim G $-$ dim H. Let $\mathfrak{G}$ and $\mathfrak{H}$ denote the Lie algebras of G and H.

1. Construction of the coordinate charts

Let $\mathfrak{P}$ denote a vector space complement of $\mathfrak{H}$ in $\mathfrak{G}$ so that $\mathfrak{G} = \mathfrak{P} \oplus \mathfrak{H}$. Define a map $\psi : \mathfrak{P}$ x $\mathfrak{H} \rightarrow G$ by $\psi(X, Y)$ =exp(X) $\cdot$ exp(Y) $=$ μ(exp(X),exp(Y)), where $\mu :$ G x G $\rightarrow$ G is the C^∞ multiplication map. Clearly ψ is a C^∞ map.

Lemma 1.1. *ψ is nonsingular at (0,0)*

Proof. Let X $\in \mathfrak{P}$, and let $\alpha(t) = \psi(tX, 0) =$ exp(tX). Then $X(e) = \psi_*(X, 0)$, which shows that $\psi_*(T_{(0,0)}(\mathfrak{P}$ x $\mathfrak{H}))$ contains $\mathfrak{P}$. A similar argument shows that $\psi_*(T_{(0,0)}(\mathfrak{P}$ x $\mathfrak{H}))$ contains $\mathfrak{H}$ and hence also $\mathfrak{P} \oplus \mathfrak{H} = \mathfrak{G} = T_e$G. It follows that $\psi_* : T_{(0,0)}(\mathfrak{P}$ x $\mathfrak{H}) \rightarrow T_eG$ is an isomorphism since dim G $=$ dim ($\mathfrak{P}$ x $\mathfrak{H}$). □

Lemma 1.2. *Let $O_1 \subset \mathfrak{P}$ and $O_2 \subset \mathfrak{H}$ be open sets containing the origins in $\mathfrak{P}$ and $\mathfrak{H}$ such that $\psi : O_1$ x $O_2 \rightarrow U$ is a diffeomorphism onto an open subset U of G containing e. Then there exists an open set $U_1 \subset O_1$ such that $\{exp(O_1)^{-1} \cdot exp(O_1)\} \cap H = \mu(exp(O_1)^{-1}, exp(O_1)) \cap H \subset exp(O_2)$.*

Proof. The set exp(O_2) is an open subset of H that contains the identity by the choice of O_1 and O_2. The group H is an imbedded submanifold of G and has the induced topology since H is closed in G (cf. Broecker-tom Dieck pp. 28-29). Choose an open set W in G that contains the identity such that W $\cap$ H $\subset$ exp(O_2). Now choose an open set $U_1 \subset O_1$ such that $\{exp(O_1)^{-1} \cdot exp(O_1)\} \subset W$. □

In the sequel we shall assume that $O_1 = U_1$ satisfies the conditions of Lemma 1.2. We continue to let O_2 and U $= \psi : O_1$ x O_2 have the meanings above. We let $\pi :$ G $\rightarrow$ G/H denote the natural projection map. Recall that π is an open map, and in particular π(U) is an open subset of G/H containing the identity coset eH.

Lemma 1.3. *Define $\varphi_0 : O_1 \rightarrow \pi(U)$ by $\varphi_0(X) =$ exp(X)H. Then φ_0 is a bijection.*

Proof. By definition U $= \psi(O_1$ x $O_2) = exp(O_1) \cdot exp(O_2)$, and therefore $\pi(U) = \pi(exp(O_1)) = \varphi_0(O_1)$ since $exp(O_2) \subset$ H. Hence $\varphi_0 : O_1 \rightarrow \pi$(U) is surjective. Now suppose that $\varphi_0(X) = \varphi_0(Y)$ or equivalently that exp(X)H $=$ exp(Y)H for some elements X,Y $\in O_1$. Then exp(X)$^{-1}\cdot$ exp(Y) $\in \{exp(O_1)^{-1} \cdot exp(O_1)\} \cap H \subset exp(O_2)$. Choose Z $\in O_2$ such that exp(X)$^{-1}\cdot$ exp(Y) $=$ exp(Z). Then $\psi(Y, 0) = exp(Y) = exp(X) \cdot exp(Z) = \psi(X, Z)$, and it follows that (Y,0) = (X,Z) since $\psi : O_1$ x $O_2 \rightarrow$ U is a diffeomorphism. Hence X = Y and φ_0 is one-one. □

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Definition of the coordinate charts

For each g $\in$ G we define $\varphi_g = L_g \circ \varphi_0$: O$_1$ $\rightarrow g \cdot \pi$(U). Each map φ_g is a bijection by Lemma 1.3, where L$_g$ denotes left multiplication by g on both G and G/H. Let U$_g$ $= \varphi_g(O_1) = g \cdot \pi(U)$. We show that the maps $\{\varphi_g : O_1 \rightarrow U_g : g \in G\}$ form a family of smoothly overlapping coordinate charts for G/H.

The set O$_1$, which is the domain of all charts $\{ \varphi_g : g \in G \}$, is an open subset of $\mathfrak{P}$, whose dimension is dim G $-$ dim H since $\mathfrak{G} = \mathfrak{P} \oplus \mathfrak{H}$. Hence it will follow that dim G/H = dim G $-$ dim H once we have proved that the charts overlap smoothly.

2. The charts overlap smoothly

Suppose $\xi \in U_g \cap U_a$ for some elements a,g $\in$ G. Then

(1) $\xi = a \cdot exp(X)H = g \cdot exp(Y)H$ for some elements X,Y of O$_1$. Moreover, $(\varphi_g^{-1} \circ \varphi_a) : \varphi_a^{-1}(U_g \cap U_a) \rightarrow \varphi_g^{-1}(U_g \cap U_a)$ is defined by $(\varphi_g^{-1} \circ \varphi_a)(X) = Y$.

Next we observe

(2) $p_0 \cdot h_0 = q_0$ for some element h_0 of H, where p$_0$ = exp(X), q$_0$ = $(a^{-1}g)\cdot$ exp(Y) and X,Y $\in$ O$_1$.

Note that $q_0 = R_{h_0}(p_0) = R_{h_0}(exp(X)) \in L_{a^{-1}g}(U)$ and X = $\varphi_a^{-1}(\xi) \in \varphi_a^{-1}(U_g \cap U_a)$, which is open in O$_1$. By (2) and the continuity of the multiplication μ : G x G $\rightarrow$ G there exists an open set A $\subset \varphi_a^{-1}(U_g \cap U_a) \subset O_1$ with X $\in$ A such that $R_{h_0}(exp(A)) \subset L_{a^{-1}g}(U)$.

Define C$^\infty$ maps $\sigma = R_{h_0} \circ exp : A \rightarrow L_{a^{-1}g}(U)$ and $\rho = \pi_1 \circ \psi^{-1} \circ L_{g^{-1}a} : L_{a^{-1}g}(U) \rightarrow O_1$, where π_1 : O$_1$ x O$_2$ $\rightarrow$ O$_1$ is the projection, and ψ : O$_1$ x O$_2$ $\rightarrow$ U is the diffeomorphism defined at the beginning of this discussion. To conclude that $(\varphi_g^{-1} \circ \varphi_a) : \varphi_a^{-1}(U_g \cap U_a) \rightarrow \varphi_g^{-1}(U_g \cap U_a)$ is C$^\infty$ it suffices to show that $\varphi_g^{-1} \circ \varphi_a = \rho \circ \sigma$ on A $\subset \varphi_a^{-1}(U_g \cap U_a) \subset O_1$.

If $X' \in$ A, then $\sigma(X') = exp(X')h_0 \in L_{a^{-1}g}(U)$ by the definition of A. Hence

(3) $exp(X')h_0 = (a^{-1}g) \cdot exp(Y_1') \cdot exp(Y_2')$ for a unique pair $(Y_1', Y_2') \in$ O$_1$ x O$_2$ = ψ^{-1}(U).

From (3) we see that $a \cdot exp(X')H = g \cdot exp(Y_1')H$ since $exp(Y_2') \in$ H. By (1) we see that $(\varphi_g^{-1} \circ \varphi_a)(X') = Y_1'$. From (3) we obtain $(\rho \circ \sigma)(X') = \rho((a^{-1}g) \cdot exp(Y_1') \cdot exp(Y_2')) = \rho(L_{a^{-1}g}\psi(Y_1', Y_2')) = Y_1'$ by the definition of ρ. Hence $\varphi_g^{-1} \circ \varphi_a = \rho \circ \sigma$ on A $\subset \varphi_a^{-1}(U_g \cap U_a) \subset O_1$.

3. The projection $\pi : G \rightarrow$ G/H is C$^\infty$

It suffices to show that $\pi : U \rightarrow \pi(U)$ is C$^\infty$, where U = ψ(O$_1$ x O$_2$) as above. Since $\pi \circ L_g = L_g \circ \pi$ for all g $\in$ G it will then follow that $\pi : L_g(U) \rightarrow L_g\pi(U)$ is C$^\infty$. This will complete the proof that $\pi : G \rightarrow$ G/H is C$^\infty$ since $L_g(U)$ is an open set containing gH for all g $\in$ G.

Using the coordinate charts $\psi : O_1$ x $O_2 \rightarrow U$ and $\varphi_0 : O_1 \rightarrow \pi(U)$ we must show that $\varphi_0^{-1} \circ \pi \circ \psi : O_1$ x $O_2 \rightarrow O_1$ is C$^\infty$. By definition $\varphi_0^{-1} \circ \pi \circ \psi(X, Y) = (\varphi_0^{-1} \circ \pi)(exp(X) \cdot exp(Y)) = (\varphi_0^{-1} \circ \pi)(exp(X)$ (since exp(Y) $\in$ H) $= X$ by the definition of φ_0. Hence $\varphi_0^{-1} \circ \pi \circ \psi : O_1$ x $O_2 \rightarrow O_1$ is projection on the first factor, which is a C$^\infty$ map.

4. The projection π : G → G/H has maximal rank

Extend the subspaces $\mathfrak{P}$ and $\mathfrak{H}$ of $\mathfrak{G} = T_eG$ to left invariant distributions on G by defining $\mathfrak{P}(g) = L_{g*}(\mathfrak{P}) \subset T_gG$ and $\mathfrak{H}(g) = L_{g*}(\mathfrak{H}) \subset T_gG$ for all g ∈ G. Then $T_gG = \mathfrak{P}(g) \oplus \mathfrak{H}(g)$ for all g ∈ G. We observe that since the elements of $\mathfrak{G}$ are left invariant vector fields on G we have $\mathfrak{P}(g) = \{X(g) : X \in \mathfrak{P}\}$ and $\mathfrak{H}(g) = \{X(g) : X \in \mathfrak{H}\}$.

Proposition 4.1. *Let π : G → G/H denote the natural projection. Then*

(a) $\pi_* : \mathfrak{P}(g) \to T_{gH}(G/H)$ is an isomorphism for all g ∈ G. In particular, $\pi_* : T_gG \to T_{gH}(G/H)$ is surjective for all g ∈ G.

(b) $\mathfrak{H}(g) = Ker\ \pi_* : T_gG \to T_{gH}(G/H)$ for all g ∈ G.

Proof. a) For g ∈ G we have the coordinate chart $\varphi_g : O_1 \to g \cdot \pi(U)$ at the point gH ∈ G/H. Let $\xi \in T_{gH}(G/H)$ be given. Since $(\varphi_g)_* : T_0O_1 \to T_{gH}(G/H)$ is an isomorphism there exists X $\in O_1 \subset \mathfrak{P}$ such that $\xi = \alpha'(0)$, where $\alpha(t) = \varphi_g(exp(tX)) = g \cdot exp(tX)H = \pi(g \cdot exp(tX))$. Hence $\xi = \pi_*((L_g)_*X(e))$, where $(L_g)_*X(e)) = X(g) \in \mathfrak{P}(g) \subset T_gG$. This proves that $\pi_* : \mathfrak{P}(g) \to T_{gH}(G/H)$ is surjective for all g ∈ G. It follows that dim $\mathfrak{P}$ = dim $\mathfrak{P}(g) \geq$ dim G/H = dim $\mathfrak{G}-$ dim $\mathfrak{H}$ = dim $\mathfrak{P}$. Hence equality must hold in the previous line, which implies that $\pi_* : \mathfrak{P}(g) \to T_{gH}(G/H)$ is an isomorphism.

(b) Since $T_gG = \mathfrak{P}(g) \oplus \mathfrak{H}(g)$ it follows from a) that dim Ker $\pi_* : T_gG \to T_{gH}(G/H) =$ dim G − dim G/H = dim G − dim $\mathfrak{P}$ = dim $\mathfrak{H}$. Hence (b) will follow for reasons of dimension when we show that $\mathfrak{H}(g) \subset Ker\ \pi_* : T_gG \to T_{gH}(G/H)$.

Let $\xi \in \mathfrak{H}(g)$. Then we may write $\xi = (L_g)_*(X(e)) = X(g)$ for some X ∈ $\mathfrak{H}$. Since $\pi_*(\xi) = (L_g)_*(\pi_*X(e))$ it suffices to prove that $\pi_*X(e)) = 0$ for every X ∈ $\mathfrak{H}$. If X ∈ $\mathfrak{H}$, then $\pi_*X(e)) = \alpha'(0)$,where $\alpha(t) = \pi(exp(tX)) = exp(tX)H$. However, $\alpha(t) = eH$ for all t since exp(tX) ∈ H for all t, and we conclude that $\alpha'(0) = 0$. □