GEOMETRY OF ANOSOV SYSTEMS: AN INTRODUCTION

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1. INTRODUCTION

At the beginning of the 1960s, D. Anosov began to study certain dynamical systems satisfying a technical condition that is now referred to by saying that the system is everywhere hyperbolic. Anosov's label for this technical condition is often translated into English as "condition U", as in [A], or "condition C", as in [AA]. This gives rise to names like "U-systems" and "C-systems" for what we shall simply call Anosov systems, following the now standard terminology introduced by Smale [S].

Anosov systems are a very rich class of dynamical systems with remarkable dynamical and geometric properties. On the dynamical side, Anosov systems illustrate concepts like hyperbolicity, structural stability, and various ergodic properties. On the geometric side, there are interesting relations with Riemannian geometry, with the theory of foliations, and with contact and symplectic geometry. These will be the main topic of the seminar. In this talk we shall start with basic definitions and examples. Later talks will discuss some foundational results concerning the integrability of the expanding and contracting distributions and structural stability, and then explore the contact and symplectic geometry of Anosov systems.

2. DISCRETE TIME SYSTEMS

Throughout M will denote a closed smooth manifold. Here is the basic definition:

Definition 2.1. A diffeomorphism $f\colon M \to M$ is Anosov if there is a continuous splitting of the tangent bundle into invariant subbundles of positive rank $TM = E^s \oplus E^u$ such that for all $n > 0$

$$||Df^n(v)|| \leq a \cdot e^{-bn}||v|| \quad \forall v \in E^s \ ,$$

$$||Df^n(v)|| \geq a \cdot e^{bn}||v|| \quad \forall v \in E^u \ ,$$

for some positive constants a and b.

Here the norms are taken with respect to some arbitrary Riemannian metric g. There is the related concept of an expanding diffeomorphism, for which the second inequality holds for all tangent vectors. This is excluded here by the assumption that both E^s and E^u have positive rank.

Remark 2.1. While the precise values of the constants a and b depend on the choice of g, the property of being Anosov does not. If the defining inequalities hold for some g, then they hold for every g (with different constants). One could also write the definitions using different constants in the two inequalities, but this seemingly more general definition is actually equivalent to the one above.

Remark 2.2. If f is Anosov, so is f^{-1}, with the roles of E^s and E^u interchanged. (This explains how to rewrite the defining inequalities for negative n.)

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Remark 2.3. The defining property of an Anosov diffeomorphism is sometimes referred to as the existence of an Anosov splitting $TM = E^s \oplus E^u$ into stable (or contracting) and unstable (or dilating) subbundles E^s and E^u respectively. This means that f is hyperbolic everywhere.

Remark 2.4. It is easy to see that when an Anosov splitting exists, it is uniquely determined by f, as the contracting and dilating subspaces have to be maximal with these properties.

We will see in later talks that the distributions E^s and E^u are actually tangent to foliations of M with smooth leaves, although the distributions are only assumed continuous. The resulting foliations are called the stable and unstable foliations of f. In general they are not smooth transversely to the leaves, but they are Hölder continuous. There are several interesting "rigidity" results which show that Anosov diffeomorphisms with smooth Anosov splitting are quite special, and can be described fairly explicitly. We shall return to this regularity discussion below.

Here are the most obvious, classical, examples of Anosov diffeomorphisms.

Example 2.1. Let $f \in SL(2, \mathbb{Z})$ be a hyperbolic element, i. e. assume that $|tr(f)| > 2$. For example, f could be $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We can think of f as a linear diffeomorphism of T^2. This is an Anosov diffeomorphism because f has two distinct real eigenvalues which are inverse to each other. Neither of them is ± 1, so the absolute value of one is in $(0, 1)$, and the absolute value of the other is in $(1, \infty)$. The eigenspaces of these eigenvalues define the Anosov splitting for f.

Hyperbolic toral automorphisms are sometimes called "cat maps", because Arnold illustrated their behaviour by drawing a stylized cat on a square, and checking how it behaves under iteration of f.

These examples illustrate some of the typical features of Anosov systems. First of all, Anosov [A] proved that for a volume-preserving Anosov diffeomeorphism the periodic points are dense. The hyperbolic toral automorphism above are volume-preserving because we assumed $det(f) = 1$. In this case we can find all the periodic points explicitly, and so see their density without using any general theorems.

Lemma 2.1. *Let $f \in SL(2, \mathbb{Z})$. Then a point $(x, y) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is periodic for f if and only if x and y are rational.*

Proof. If x and y are rational, then the coordinates of $f^n(x, y)$ are also rational for all n, because f is an integer matrix. Moreover, if x and y can be written with common denominator q, then so can the coordinates of $f^n(x, y)$. As there are only finitely many (in fact q^2) points whose denominators can be taken to be q, it follows that the sequence of points $f^n(x, y)$ is eventually periodic. But f is invertible, so (x, y) is periodic.

Conversely, assume (x, y) is periodic with period $n \geq 1$. Then the equation $f^n(x, y) = (x, y)$ is a system of linear equations for x and y with integral coefficients. It follows that x and y are rational. □

Anosov [A] also proved that volume-preserving Anosov diffeomorphisms are ergodic. A related property is that of topological transitivity.

Definition 2.2. A diffeomorphism $f \colon M \to M$ is topologically transitive if for any two open subsets $U, V \subset M$ there is an $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

Lemma 2.2. *Let $f \in SL(2, \mathbb{Z})$ be such that $|tr(f)| > 2$. Then $f \colon T^2 \to T^2$ is topologically transitive.*

Proof. First note that the eigenvalues of f are irrational. (They are rational if and only if $(tr(f))^2 - 4$ is a perfect square, which is seen to be impossible using elementary divisibility properties.) Therefore each leaf of the stable or the unstable foliation is dense.

Consider two open sets $U, V \subset T^2$. If they intersect, there is nothing to prove. So we assume they are disjoint. Take points with rational coordinates $P \in U$ and $Q \in V$. By the previous lemma, P and Q are periodic, so by replacing f by a suitable power, we may assume without loss of generality that P and Q are fixed points.

Let L^u_P be the leaf of the unstable foliation through P, and L^s_Q the leaf of the stable foliation through Q. As both of these leaves are dense, they certainly intersect. Let R be a point of intersection. The stable and unstable foliations are invariant under f, and because P and Q are fixed points, the leaves L^u_P and L^s_Q are preserved. Under forward iteration of f, the images of R converge to Q, whereas under backward iteration the images of R converge to P. Thus a suitable power f^n of f satisfies $f^n(U) \cap V \neq \emptyset$. □

The following generalization of Example 2.1 is due to Smale [S, AA].

Example 2.2. Let G be a simply connected nilpontent Lie group. Denote by $Aut(G)$ the group of continuous group automorphisms, and by $Aff(G)$ the semidirect product of $Aut(G)$ and G given by the action of $Aut(G)$ on G. Sometimes there are elements $\Phi \in Aut(G)$ which commute with some lattice $\Gamma \subset Aff(G)$. Whenever this happens, Φ descends to a diffeomorphism φ of the infranil manifold G/Γ. A sufficient condition for φ to be Anosov is that the derivative of Φ, considered as an automorphism of the Lie algebra $\mathfrak{g}$ of G, is a hyperbolic linear map.

It follows from the theory of Lie algebras that the existence of a hyperbolic automorphism of $\mathfrak{g}$ implies that $\mathfrak{g}$ is nilpotent, see [S] p. 761. The case of an Abelian Lie algebra is the multidimensional analog of Example 2.1.

Let us now consider a relationship between Anosov diffeomorphisms and symplectic geometry.

Lemma 2.3. *Let (V, ω) be a closed symplectic manifold, and $f\colon V \to V$ an Anosov symplectomorphism. Then E^s and E^u are Lagrangian with respect to ω.*

Proof. Suppose $v, w \in E^s$. Then

$$\omega(v,w) = (f^*\omega)(v,w) = \omega(Df(v), Df(w)) = \ldots = \omega(Df^n(v), Df^n(w)) .$$

Using the auxiliary metric g, we find that there is a constant c such that

$$|\omega(v,w)| \leq c \cdot ||\omega|| \cdot ||Df^n(v)|| \cdot ||Df^n(w)|| \leq c \cdot ||\omega|| \cdot a^2 \cdot e^{-2bn} \cdot ||v|| \cdot ||w|| .$$

Letting n go to infinity, the right-hand-side becomes arbitrarily small. Therefore $\omega(v,w) = 0$, and E^s is ω-isotropic. By the same argument with f^{-1} replacing f we conclude that E^u is also ω-isotropic. As the two distributions are complementary, they must be equidimensional and Lagrangian. □

This Lemma shows that the stable and unstable foliations of an Anosov symplectomorphism define a Künneth structure, as studied by M. Hamilton, among others. For Künneth structures one has an intrinsically defined torsion-free affine connection, called the Künneth connection, for which ω and the two Lagrangian foliations are parallel. In the case of an Anosov diffeomorphism f, which does not necessarily preserve a symplectic form, there is a distinguished connection for which the stable and unstable foliations are parallel. This is called the Kanai connection, and we shall consider it in one of the later talks in this seminar. In the case of an Anosov symplectomorphism, the Künneth and Kanai connections agree.

While it is not clear whether the Künneth connection of an arbitrary Künneth structure has any good completeness properties, for the Kanai connection of an Anosov diffeomorphism one can show, by an argument like the one in the proof of Lemma 2.3, that it is complete when restricted to the leaves of the stable and unstable foliations.

It is now known that there are so-called exotic Anosov diffeomorphisms, which are not topologically transitive, and also have other unusual properties. Nevertheless, already in the 1960s conjectures began to appear which suggested that if one assumes that the stable and unstable foliations are smooth (plus perhaps some technical conditions), then any Anosov diffeomorphism satisfying this regularity property should be an infranil automorphism as in Example 2.2. The most comprehensive result in this direction is the following:

Theorem 2.1 (Benoist–Labourie [BL]). *Let $f\colon M \to M$ be an Anosov diffeomorphism such that*

(a) *f preserves an affine connection on M,*
(b) *f is topologically transitive, and*
(c) *the Anosov splitting is smooth of class C^∞.*

Then f is C^∞-conjugate to a hyperbolic infranil automorphism[1].

This is one of the main theorems on Anosov systems which we will *not* cover in this seminar. The proof in [BL] relies on the proof of a related theorem for flows in [BFL], which we shall discuss below.

The following consequence of Theorem 2.1 is stated without proof in [BL].

Corollary 2.1. *Let $f\colon V \to V$ be an Anosov symplectomorphism of (V, ω) for which the Anosov splitting is smooth. Then f is C^∞-conjugate to a hyperbolic infranil automorphism.*

Proof. As f preserves ω and the Lagrangian foliations given by the stable and unstable subbundles, see Lemma 2.3, we conclude that it preserves the Künneth connection, so we have assumption (a) in Theorem 2.1.

To check condition (b), note that f is volume-preserving. It is a theorem of Anosov [A] that a volume-preserving Anosov diffeomorphism is ergodic with respect to the invariant measure given by the volume form. This implies topological transitivity. □

It was pointed out in [BL] that for every hyperbolic infranil automorphism φ of G/Γ there is a left-invariant Φ-invariant connection on G which is flat, torsion-free, and complete. This descends to G/Γ, so that, *a posteriori*, all topologically transitive Anosov diffeomorphisms with smooth Anosov splitting preserving some affine connection actually preserve a flat, torsion-free, complete connection.

3. Continuous time systems

We now consider the case of a flow φ_t defined by a time-independent smooth vector field X on a smooth closed manifold M. Here is the basic definition:

Definition 3.1. A vector field X on M is Anosov if there is a continuous splitting of the tangent bundle into invariant subbundles of positive rank $TM = \mathbb{R}X \oplus E^s \oplus E^u$ such that for all $t > 0$

$$||D\varphi_t(v)|| \leq a \cdot e^{-bt}||v|| \quad \forall v \in E^s ,$$
$$||D\varphi_t(v)|| \geq a \cdot e^{bt}||v|| \quad \forall v \in E^u ,$$

for some positive constants a and b.

[1] cf. Example 2.2

Here again the norms are taken with respect to some arbitrary Riemannian metric g.

Remark 3.1. While the precise values of the constants a and b depend on the choice of g, the property of being Anosov does not. If the defining inequalities hold for some g, then they hold for every g (with different constants). One could also write the definitions using different constants in the two inequalities, but this seemingly more general definition is actually equivalent to the one above.

Remark 3.2. If X is Anosov, so is $-X$, with the roles of E^s and E^u interchanged. (This explains how to rewrite the defining inequalities for negative t.)

Remark 3.3. The defining property of an Anosov vector field or flow is sometimes referred to as the existence of an Anosov splitting $TM = \mathbb{R}X \oplus E^s \oplus E^u$ which splits the normal bundle of the orbits into stable (or contracting) and unstable (or dilating) subbundles E^s and E^u respectively. This means that φ_t is everywhere hyperbolic normally to the flow.

Remark 3.4. It is easy to see that when an Anosov splitting exists, it is uniquely determined by X, as the contracting and dilating subspaces have to be maximal with these properties.

The above remarks are all in parallel with the corresponding remarks for discrete time systems. The following remark does not have such a parallel:

Remark 3.5. Because M is compact, every rescaling of an Anosov vector field is also Anosov. Thus the Anosov property is a property of the underlying 1-dimensional foliation.

If f is an Anosov diffeomorphism, so is f^n for all $n \neq 0$, and one could think of this as a reparametrization of time. However, this really corresponds to the trivial case of constant rescalings, whereas the interesting instances of the above remark are non-constant rescalings.

We will see in later talks that the distributions E^s and E^u, and also $\mathbb{R}X \oplus E^s$ and $\mathbb{R}X \oplus E^u$, are actually tangent to foliations of M with smooth leaves, although the distributions are only assumed continuous. The resulting foliations are called the strongly, respectively the weakly, stable and unstable foliations of X or φ_t. In general these foliations are not smooth transversely to the leaves, but they are Hölder continuous. There are several interesting "rigidity" results which show that Anosov flows with smooth Anosov splitting are quite special, and can be described fairly explicitly. We shall return to this regularity issue after mentioning the standard examples.

Example 3.1. If f is any Anosov diffeomorphism of M, then its suspension defines an Anosov flow φ_t on the mapping torus M_f. In this case the strongly stable and unstable distributions of φ_t restrict to each fiber of the mapping torus as the stable and unstable distributions for f.

Example 3.2. Let N be a closed Riemannian manifold of negative sectional curvature. Then the geodesic spray on the unit tangent bundle $M = S(N)$ defines an Anosov vector field X, whose strongly stable and unstable foliations are the so-called horocycle foliations.

This was one of the motivating examples for Anosov's original work [A]. We shall consider the examples of geodesic flows in some detail in the next talk.

Whenever we are given an Anosov vector field on M, we can associate with it a continuous 1-form γ defined by $\gamma(X) = 1$ and $\ker(\gamma) = E^s \oplus E^u$. The form γ is not smooth in general, and even when it is, there is no reason why it should be of constant class. However, when this happens, then one obtains more information. This is well-known in the two extreme cases, when γ defines a foliation or a contact structure.

In the case when the kernel of γ defines a foliation, the Anosov flow tends to be, up to suitable rescaling, the suspension of an Anosov diffeomorphism. Results in this direction have been proved by Plante, and we shall discuss his work later in this seminar.

In the other extreme case one makes the following definition:

Definition 3.2. A vector field X is said to be Anosov-contact if it is Anosov and the one-form γ is a smooth contact form.

We have the following elementary observation[2]:

Lemma 3.1. *If X is Anosov-contact, then it is the Reeb vector field of γ, and E^s and E^u are Lagrangian for the symplectic structure defined on $ker(\gamma)$ by $d\gamma$.*

Proof. By the definition of the stable and unstable subbundles, they are invariant under the flow of X. Thus this flow preserves the kernel of γ. It also preserves X tautologically, and thus we conclude $L_X\gamma = 0$. This, together with $\gamma(X) = 1$, implies $i_X d\gamma = 0$, and so X is the Reeb vector field of γ.

The second claim is proved by the same argument as in the proof of Lemma 2.3. Let φ_t be the flow of X. Suppose $v, w \in E^s$. Then, using $L_X\gamma = 0$, we find

$$d\gamma(v, w) = (\varphi_t^* d\gamma)(v, w) = d\gamma(D\varphi_t(v), D\varphi_t(w)) .$$

Using the auxiliary metric g, we find that there is a constant c such that

$$|d\gamma(v, w)| \leq c \cdot ||d\gamma|| \cdot ||D\varphi_t(v)|| \cdot ||D\varphi_t(w)|| \leq c \cdot ||d\gamma|| \cdot a^2 \cdot e^{-2bt} \cdot ||v|| \cdot ||w|| .$$

Letting t go to infinity, the right-hand-side becomes arbitrarily small. Therefore $d\gamma(v, w) = 0$, and E^s is isotropic for $d\gamma$. By the same argument, but letting t go to $-\infty$, we conclude that E^u is also isotropic. As the two distributions are complementary in $ker(\gamma)$, they must be equidimensional and Lagrangian. □

Here is the theorem characterizing Anosov-contact flows in terms of standard examples:

Theorem 3.1 (Benoist–Foulon–Labourie [BFL]). *Let X be an Anosov-contact vector field with C^∞ smooth Anosov splitting on V^{2n-1}. Then there is a unique cohomology class $[\alpha] \in H^1(V; \mathbb{R})$ and a representing closed 1-form α such that $1 + \alpha(X) > 0$, and defining $Y = X/(1 + \alpha(X))$ we have:*

(i) *if $n \geq 3$, then there is a finite covering of V on which the lift of Y is C^∞-conjugate to the geodesic flow of a locally symmetric space with negative sectional curvature;*

(ii) *if $n = 2$, then there is a finite covering of V on which the lift of Y is C^∞-conjugate to the lift to a finite covering of the geodesic flow of a hyperbolic surface*[3].

This is the second main theorem about Anosov systems which we do *not* cover in this seminar. The proof uses a version of the Kanai connection for Anosov flows (which we will discuss later) in order to set up a situation in which one can appeal to Gromov's "open orbit theorem". After that, the bulk of the proof goes through the theory of algebraic groups to find that (V, X) is indeed diffeomorphic to one of the standard homogeneous examples. The proof of Theorem 2.1 follows the same approach, and in fact essentially appeals to the proof of Theorem 3.1.

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[2] which I missed during the seminar talk

[3] This second part had been proved earlier by Ghys.

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