Mathematisches Institut
Universität München

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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 9

1. Let M be a compact oriented differentiable manifold with boundary $\partial M \neq \emptyset$. Show that there is no retraction of M onto ∂M, i.e. no smooth map $r : M \to \partial M$ with $r|_{\partial M} = Id_{\partial M}$.
Hint: Try to use Stokes' Theorem.

2. Let M^m, N^n be differentiable manifolds. Smooth maps $f, g : M \to N$ are called *homotopic* if there exists a smooth map $H : M \times [0,1] \to N$ such that $H(-,0) = f$, $H(-,1) = g$. Suppose M is closed and orientable, $\omega \in \Omega^m(N)$ a closed m-form on N and f, g homotopic. Show that

$$\int_M f^*\omega = \int_M g^*\omega.$$

3. Let M be a connected differentiable manifold and $\alpha \in \Omega^1(M)$ a 1-form. For a smooth map $\phi : S^1 \to M$ define

$$\int_\phi \alpha := \int_{S^1} \phi^*\alpha.$$

Show that

$$\alpha \text{ is exact} \Longleftrightarrow \int_\phi \alpha = 0 \quad \text{for all} \quad \phi : S^1 \to M.$$

4. a) A differentiable manifold M is called *simply-connected* if every smooth map $\phi : S^1 \to M$ is homotopic to a constant map. Suppose M is simply-connected. Prove that $H^1_{DR}(M) = 0$.

b) Let M, N be smooth manifolds and $f : N \to M$ differentiable. Prove that the *cup-product* $\cup : H^k_{DR}(M) \times H^l_{DR}(M) \to H^{k+l}_{DR}(M)$ given by

$$[\omega] \cup [\eta] = [\omega \wedge \eta]$$

is well-defined and that $f^*([\omega] \cup [\eta]) = f^*[\omega] \cup f^*[\eta]$.

Please hand in your solutions on Friday, the 22nd of December, by 4pm.