Mathematisches Institut
Universität München

Jonathan Bowden
Office hour: Wed 2-3 pm, Room 308
WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 8

1. Let M be a differentiable manifold of dimension n. Prove that the tangent bundle TM, considered as a $2n$-dimensional smooth manifold, is always orientable.

2. Let M be an oriented differentiable manifold of dimension n and $f : M \to M$ a diffeomorphism. The map f is called *orientation-preserving at* $p \in M$ if the Jacobian of $D_p f$ has positive determinant with respect to oriented charts around p and $f(p)$. The map f is called *orientation-preserving on* M if it is orientation-preserving at every point.

a) Suppose M is connected. Show that f is orientation-preserving on M if and only if it is orientation-preserving at a single point $p \in M$.

b) Suppose M is connected, $f : M \to M$ a diffeomorphism and $\omega \in \Omega^n(M)$ a nowhere vanishing section of $\Lambda^n T^*M$. Show that $f^*\omega = g\omega$ for a non-vanishing function $g \in C^\infty(M)$. Prove that f is orientation-preserving on M if and only if $g > 0$ everywhere.

3. a) Let $f : S^2 \to S^2, x \mapsto -x$, be the antipodal map. Show that f is not orientation-preserving on S^2.

b) Prove that $\mathbb{R}P^2$ is not orientable.

4. Let M, N be differentiable manifolds, $f : M \to N$ smooth and α a differential form on N.

a) Show that $d(f^*\alpha) = f^* d\alpha$.

b) Let X be a vector field on M and ω a differential form on M. Show that $(i_X\omega)(p)$ at $p \in M$ depends on X only through $X(p)$. Prove that

$$i_X(f^*\alpha)(p) = f^*(i_{D_pf(X)}\alpha)(p).$$

Please hand in your solutions on Friday, the 15th of December, by 4pm.