Mathematisches Institut
Universität München

Jonathan Bowden
Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 7

1. Let $\pi : E \to B$ be a differentiable vector bundle of rank n over a differentiable manifold, and $f : M \to B$ a differentiable mapping from a manifold M to B. Let

$$f^*E = \{(p, v) \in M \times E \mid f(p) = \pi(v)\}.$$

a) Prove that with the projection onto the first factor, f^*E is a differentiable vector bundle over M of rank n. The bundle $f^*E \to M$ is called the *pull-back bundle* under f. Compute the transition functions for f^*E in terms of the transition functions for E.

c) Prove for vector bundles E, F over B that $f^*(E \oplus F) \cong f^*E \oplus f^*F$, $f^*(E \otimes F) \cong f^*E \otimes f^*F$ and $f^*(\Lambda^k E) \cong \Lambda^k(f^*E)$ for all k.

2. Let V be a vector space of dimension $n > 0$. An *orientation* of V consists of an equivalence class of ordered bases under basis changes whose matrices have positive determinant.

a) Show that V has precisely two orientations and that an orientation on V is equivalent to an orientation on $\Lambda^n(V^*)$. A positively oriented basis element of $\Lambda^n(V^*)$ is called a *volume form* on V.

b) A 2-form $\omega \in \Lambda^2(V^*)$ is called *non-degenerate* or *symplectic* if for each non-zero vector $x \in V$ there exists a $y \in V$ such that $\omega(x, y) \neq 0$. Prove that symplectic forms exist only on even dimensional vector spaces. If V is a vector space of dimension $2k$ and ω a 2-form, prove that ω is symplectic if and only if ω^k is a volume form on V.

3. Let M be a smooth manifold, α, β differential forms and X, Y vector fields. Prove the following formulas:

$$\begin{aligned} L_X(\alpha \wedge \beta) &= (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta). \\ i_{[X,Y]}\alpha &= L_X i_Y \alpha - i_Y L_X \alpha. \end{aligned}$$

4. Let M be a smooth manifold and $\omega \in \Omega^2(M)$ a 2-form. Show that for vector fields X, Y, Z

$$\begin{aligned} d\omega(X, Y, Z) &= L_X(\omega(Y, Z)) + L_Y(\omega(Z, X)) + L_Z(\omega(X, Y)) \\ &\quad -\omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y). \end{aligned}$$

Please hand in your solutions on Friday, the 8th of December, by 4pm.