Mathematisches Institut
Universität München

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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 6

1. a) Let V be a real vector space of dimension n and $m \geq 1$. Prove that there is a canonical isomorphism $(\Lambda^m V)^* \cong \Lambda^m(V^*)$.

Let $\omega \in \Lambda^2(V^*)$ be a 2-form, i.e. an alternating bilinear map $V \times V \to \mathbb{R}$. The matrix of ω in a basis $e_1, ..., e_n$ of V is $\omega_{ij} = \omega(e_i, e_j)$.

b) Let V be 2-dimensional and ω not identical to 0. Show that there exists a basis e_1, e_2 of V such that the matrix of ω has the form

$$\omega_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

c) Let V have arbitrary dimension n. Show that there exists a basis $e_1, ..., e_n$ of V and a number k with $2k \leq n$ such that

$$\omega = \sum_{i=1}^{k} \alpha_{2i-1} \wedge \alpha_{2i},$$

where $\alpha_1, ..., \alpha_n$ denotes the dual basis of V^*. (For the proof, you can consider the subspace $W' = \{v \in V \mid \omega(v, w) = 0 \ \forall w \in W\}$ for a subspace W of V.)

2. Let V be an n-dimensional vector space. A 2-form $\omega \in \Lambda^2(V^*)$ is called *decomposable* if there exist 1-forms α, β such that $\omega = \alpha \wedge \beta$. Using Ex.1, show that ω is decomposable if and only if $\omega \wedge \omega = 0$. If the dimension $n \geq 4$, find a 2-form which is not decomposable.

3. a) Let K be a field and V, W finite-dimensional K-vector spaces. Prove that there is a canonical isomorphism $V^* \otimes W \cong Hom_K(V, W)$.

b) Let M be a manifold. A *line bundle* $L \to M$ is a rank 1 vector bundle over M. Prove that the set $\mathcal{L}$ of isomorphism classes of line bundles over M is an abelian group with multiplication given by tensor product.

4. a) Show that the vector bundle $TS^n \oplus \epsilon^1$ is trivial for all $n \geq 0$, with ϵ^1 the trivial line bundle over the sphere S^n.

b) Show that the tangent bundle of every odd-dimensional sphere splits off a trivial line bundle, i.e. $TS^{2n-1} = \epsilon^1 \oplus E$ for some vector bundle E.

c) Show that $S^{2n-1} \times S^k$ has trivial tangent bundle for all $n, k \geq 1$.

Please hand in your solutions on Friday, the 1st of December, by 4pm.