Mathematisches Institut
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Office hour: Wed 2-3 pm, Room 308 WS 06/07

**Geometry of Manifolds I (Prof. Kotschick)**
**Problem sheet 5**

1. a) Let $(U, \phi)$ be a chart of an $n$-dimensional differentiable manifold $M$. The coordinates of a point $p \in U$ are $(x_1(p), ..., x_n(p)) = \phi(p) \in \mathbb{R}^n$. Let $e_1, ..., e_n$ denote the *coordinate vector fields* $e_i = \partial/\partial x_i$, defined by their actions on smooth functions on $U$. Show that $e_1, ..., e_n$ form for each $p \in U$ a basis of the tangent space $T_pM$ and that $[e_i, e_j] = 0$ for all $i, j$.
b) Let $M$ be a differentiable manifold and $f, g : M \to \mathbb{R}$ smooth functions. Show that

$$[fX, gY] = f(L_X g)Y - g(L_Y f)X + fg[X, Y],$$

for all vector fields $X, Y$ on $M$. Calculate as an example the commutator of the vector fields $V = x^2 y e_1 + e_2$ and $W = x e_1 - y^2 e_2$ on $\mathbb{R}^2$ with coordinates $(x, y)$ and $e_1 = \partial/\partial x, e_2 = \partial/\partial y$.

2. Let $M$ be a differentiable manifold. For $\psi : M \to N$ a diffeomorphism and $X$ a vector field on $M$, we define the "push-forward" of $X$ as $(\psi_* X)_{\psi(p)} = (D_p\psi)(X_p)$.
a) For $X$ and $Y$ vector fields on $M$ show that

$$\psi_*[X, Y] = [\psi_* X, \psi_* Y].$$

b) Let $\Phi_t$ be the flow of a vector field $X$ and let $\alpha : M \to N$ be a diffeomorphism. Show that the flow of $\alpha_* X$ is given by $\alpha \circ \Phi_t \circ \alpha^{-1}$.

3. Let $M$ be a differentiable manifold and $X, Y$ complete vector fields. Denote the flows by $\Phi_t^X$ and $\Phi_s^Y$, respectively. Show that the commutator vanishes if and only if the flows commute, i.e.

$$[X, Y] \equiv 0 \Longleftrightarrow \Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X \quad \forall t, s \in \mathbb{R}.$$

Hint: For the '$\Leftarrow$' direction use 2 b) above and the fact that $[X, Y] = L_X Y$. For the other direction you should consider for fixed $p \in M$ the curve $c(t) : (-\epsilon, \epsilon) \to T_pM$

$$c(t) = ((\Phi_t^X)_* Y)_p$$

and show that this is constant for all $t$.
Remark: The proof of the fact that the flows $\Phi_t^X$ and $\Phi_s^Y$ commutes if and only if the Lie bracket $[X,Y]=0$ is of a purely local nature so the same conclusion holds without the completeness assumption. Note that in this case the flows now are only defined for certain values of $(t,x),(s,y)\in\mathbb{R}\times M$ so the statement $\Phi_t^X\circ\Phi_s^Y=\Phi_s^Y\circ\Phi_t^X$ will on make sense locally, ie if one restricts the flows to certain neibourhoods of a chosen point, then for sufficiently small $t,s$, the flows and the compositions will be well defined. Thus the conlusion will hold in some neibourhood of every poin $x\in M$. You may use this more general result in the final problem.

4. Let $M$ be a 2-dimensional differentiable manifold. Suppose there are vector fields $e_1,e_2$ defined in the neighbourhood of a point $p\in M$ which are everywhere linearly independent and such that the commutator $[e_1,e_2]$ vanishes identically. Show that there is a chart $(U,\phi)$ around $p$ such that $e_1,e_2$ restricted to $U$ are the associated coordinate vector fields.

Please hand in your solutions on Thursday, the 24th of November, by 4pm.