Mathematisches Institut
Universität München

Jonathan Bowden
Office hour: Wed 2-3 pm, Room 308 WS 06/07

## Geometry of Manifolds I (Prof. Kotschick)
## Problem sheet 3

1. Recall that a smooth map $\eta$ between vector two vector bundles $E, E'$ over a manifold $M$ is called a bundle map if the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\eta"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

and $\eta$ is linear when restricted to fibers. Assuming that the rank of $\eta$ is constant show that
a) $Ker(\eta) = \{v \in E \mid \eta(v) = 0\}$ is a subbundle of $E$.
b) $Im(\eta) = \{\eta(v) \mid v \in E\}$ is a subbundle of $E'$.
If this seems hard think what you would do if instead of bundles we had vector spaces, this is the case when $M = pt$ is a single point.

2. a) Let $M = \mathbb{R}^2$ be n dimensional Euclidean space the. This is covered by one chart so its tangent bundle is trivial, ie $T\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$. For this tangent bundle define bundle maps $\eta : T\mathbb{R}^2 \to T\mathbb{R}^2$ which are NOT of constant rank. Are the kernel/respectively the image of your maps subbundles of $T\mathbb{R}^2$? Why/why not?
b) Let $S^n \subset \mathbb{R}^{n+1}$ and consider the "height" function $h : S^n \to \mathbb{R}$ defined by $h(x_1, ..., x_{n+1}) = x_{n+1}$. Compute $Dh$ in terms of the stereographic coordinates from sheet 1 and determine at what points it is surjective and what points it is not.

3. Let $F \subset E$ be a subbundle, where $E$ is a vector bundle over $M$.

Show that $E/F$ is a vector bundle over $M$ where $E/F = E/\sim$ where $v \sim v'$ if and only if $\pi(v) = \pi(v')$ and there is a $w \in F$ so that $v - v' = w$. Deduce that there is an exact sequence of bundle maps $0 \to Ker(\eta) \to E \to E'/Im(\eta) \to 0$.

4.Let E be a rank $n$ vector bundle over $M$ endowed with a metric $g$.
a) Show that for every $x \in M$ there is a neighbourhood of $U$ of $x$ and $n$ sections, $s_i$ of E, which are defined on $U$ so that for each $x \in U$ $\{s_1(x), s_2(x)....s_n(x)\}$ form an othonormal basis for the fibre over $x$ with respect to the metric $g$.

b) Let $g$ be a metric on E for $F \subset E$ a subbundle of E, define $F^{\perp} := \{v \in E \mid g(v, w) = 0, \text{ for all } w \in F \text{ with } \pi(w) = \pi(v)\}$. Show using part a) that $F^{\perp}$ is a vector bundle over M. Furthermore show that $E/F \cong F^{\perp}$ as bundles.

5. Using the following description of the tangent bundle of a sphere as $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid x.v = 0\}$, construct a non-vanishing section on $TS^{2n+1}$ for $n \geq 0$.
For extra credit show that $TS^3 \cong S^3 \times \mathbb{R}^3$.
(Hint: since $n$ is odd you may identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$).

Please hand in your solutions on Friday the 10th of November by 4 PM, in the appropriate box.