Mathematisches Institut
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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 2

1. Let F be a differentiable manifold and $\phi : F \to F$ a diffeomorphism. Consider the topological space $F \times \mathbb{R}$ with the equivalence relation $\sim$ given by $(x,t) \sim (y,s) \Leftrightarrow (\phi^n(x), t+n) = (y,s)$ for some $n \in \mathbb{Z}$

Let $M = (F \times \mathbb{R})/_\sim$ denote the quotient space. This is called the *mapping torus* of ϕ.

a) Show that $\sim$ is an equivalence relation. The quotient is clearly Hausdorff and second countable. Define explicitly a differentiable structure on M, induced from the structure on F.

Important examples of mapping tori are the 2-torus T^2 ($F = S^1, \phi(x) = x$), the Klein bottle K^2 ($F = S^1, \phi(x) = -x$), the Möbius band ($F = \mathbb{R}, \phi(x) = -x$) and the twisted 2-sphere bundle over the circle, $S^2 \tilde{\times} S^1$ ($F = S^2, \phi(x) = -x$).

b) Show that a line bundle (i.e. a vector bundle of rank 1) over a manifold X is trivial if and only if it admits a nowhere vanishing section. Show that the tangent bundle of S^1 is trivial. Show that the Möbius band is a line bundle over the circle S^1 and prove that it is non-trivial.

2. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus 0$ by $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus 0 : x = \lambda y$. Real projective n-space is the space of equivalence classes

$$\mathbb{R}P^n := (\mathbb{R}^{n+1} \setminus 0)/_\sim$$

with the quotient topology. The equivalence class of a point $(x_0, ..., x_n)$ is denoted by $[x_0 : ... : x_n]$.

Show that $\mathbb{R}P^n$ is a compact n-dimensional differentiable manifold (you can use the sets $U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0\}$ for $i = 0, ..., n$ as charts). Let

$$\pi : S^n \to \mathbb{R}P^n, (x_0, ..., x_n) \mapsto [x_0 : ... : x_n]$$

be the restriction of the canonical projection. Show that π is surjective and that the preimage of every $p \in \mathbb{R}P^n$ consists of two points (i.e., S^n is a two-sheeted covering of $\mathbb{R}P^n$).

3. Similarly one defines complex projective n-space as

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus 0)/_\sim, \quad \text{where} \quad z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C} \setminus 0 : z = \lambda w.$$

As above, one can show that this is a compact 2n-dimensional differentiable manifold. Consider the restriction of the canonical projection to $S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Show that this is a differentiable map, whose derivative at every point is surjective. What is the preimage of a point in $\mathbb{C}P^n$?

4 . Consider the case $n = 1$, from the previous exercise. This is a smooth manifold $\mathbb{C}P^1 = \{[z_1 : z_2] \mid z_i \in \mathbb{C}\}$.

a) Write down the charts on $\mathbb{C}P^1$ With respect to the covering $U_i = \{[x_1 : z_2] \mid x_i \neq 0\}$ for $i = 1, 2$, do this first in complex then is real co-ordinates.

b) Use this to construct a diffeomorphism of $\mathbb{C}P^1$ to S^2. (Hint: cf. problem 4 from last week)

So the map from problem three in the case $n = 1$ is a map $S^3 \to S^2$ which is called the 'Hopf fibration'. It is a nice exercise in mental gymnastics to try to visualise what the map actually looks like, ie what the pre-images of points in S^2 look like in a chart $S^3/\{N\}$ of S^3. NOTE: You don't have to do this for credit!!!

Please hand in your solutions on Friday the 3rd of November by 11 AM, in the appropriate box.