Mathematisches Institut
Universität München

Jonathan Bowden
Office hour: Wed 2-3 pm, Room 308
WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 15

1. For the following vector fields on $\mathbb{R}^2$ sketch the flow lines and determine the index of the singularity at $(x, y) = (0, 0)$.

$$a)\, V(x, y) = (-y, x) \quad b)\, V(x, y) = (x, -y) \quad c)\, V(x, y) = (x^2 - y^2, 2xy).$$

2. For every integer $n \in \mathbb{Z}$ construct a vector field on $\mathbb{C} \cong \mathbb{R}^2$ which has a singularity of index n at $z = 0$. Sketch the flow lines for $n = \pm 2$ and $n = \pm 3$.

3. Find a smooth vector field on S^2 with precisely one zero.

4. Let M, N be smooth closed oriented manifolds of dimension n and $\phi : M \to N$ a differentiable map.
a) Prove that $\phi^* : H^n_{dR}(N) \to H^n_{dR}(M)$ is multiplication by some real number d. (This number is called the *degree* of ϕ and is in fact always an integer, but you do not have to prove this.)
b) Find for every integer $d \in \mathbb{Z}$ a smooth map $\phi_d : S^n \to S^n$ with degree d.

5. Let Σ_g be an orientable surface of genus g and $\Gamma \subset \Sigma_g$ a finite connected graph. Let v and e denote the number of vertices and edges of Γ and f the number of connected components of $\Sigma_g \setminus \Gamma$. Prove that

$$v - e + f = \chi(\Sigma_g),$$

if all connected components of $\Sigma_g \setminus \Gamma$ are diffeomorphic to disks.

6. Let F be a not necessarily orientable surface with a metric g on the tangent bundle TF and a compatible connection ∇. Give a definition of the Gauss curvature $\mathcal{K} \in C^\infty(F)$ which reduces to the definition from the lecture in case F is oriented.

Please hand in your solutions in the first lecture of Geometry of manifolds II, on Tuesday, the 17th of April 2007.