Mathematisches Institut
Universität München

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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 14

1. Let V be a finite-dimensional vector space with a scalar product $\langle , \rangle$. For $\phi \in End\, V$ define its *adjoint* ϕ^* via $\langle \phi^* v, v' \rangle = \langle v, \phi v' \rangle$ for all $v, v' \in V$. The endomorphism ϕ is called *symmetric* if $\phi^* = \phi$ and *skew-symmetric* if $\phi^* = -\phi$.

a) Prove that $(\phi^*)^* = \phi$ and that there is a direct sum decomposition $End\, V = V_+ \oplus V_-$, where V_+, V_- denote the vector space of symmetric and skew-symmetric endomorphisms.

b) Let $E \to B$ be a vector bundle with a metric $\langle , \rangle$. Do the same construction as above fibrewise for the fibres of $End\, E$. Show in particular that there are vector bundles E_+, E_- with $End\, E = E_+ \oplus E_-$ and that the sections of E_- are the skew-symmetric endomorphisms of E defined in the lecture.

2. Let $E \to B$ be a vector bundle with a metric $\langle , \rangle$. We use the same notation as above. Show that $E_- \cong \Lambda^2 E$, and conclude that if E is oriented and of rank 2, then E_- is a trivial line bundle.

3. Let $\pi : E \to M$ be an oriented rank 2 vector bundle over a manifold M and suppose $\langle\, , \,\rangle_0$ and $\langle\, , \,\rangle_1$ are two metrics on E. Then the vector bundle

$$E \times [0,1] \to M \times [0,1], (v,t) \mapsto (\pi(v), t),$$

has a metric $\langle\, , \,\rangle$ which on $E \times \{t\} \to M \times \{t\}$ is given by

$$\langle\, , \,\rangle_t = (1-t)\langle\, , \,\rangle_0 + t\langle\, , \,\rangle_1.$$

Use this construction and Ex. 1 on sheet 10 to show that the Euler class $e(E, \langle , \rangle) \in H^2_{DR}(M)$ of an oriented rank 2 vector bundle is independent of the chosen metric.

4. Let $E \to M$ be an oriented rank 2 vector bundle. Prove that if E admits a nowhere vanishing smooth section then the Euler class $e(E)$ vanishes. Use this to show that every smooth vector field on S^2 has a zero.

Please hand in your solutions on Friday, the 9th of February, by 4pm.