Mathematisches Institut
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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 13

1. Let M be a differentiable manifold and $E \to M$ the trivial rank 1 vector bundle with a fixed trivialization. Two connections ∇, ∇' are called *gauge-equivalent* if there exists a smooth map $g : M \to \mathbb{R}^*$ such that $\nabla' s = g^{-1} \cdot \nabla(g \cdot s)$ for all sections s of E. Let ∇ be a flat connection. Then in the trivialization of E, the connection 1-form of ∇ is a closed differential form $\alpha \in \Omega^1(M)$. Conversely, every closed 1-form α on M determines a flat connection ∇^α. Show that:

$$\nabla^\alpha,\ \nabla^{\alpha'} \text{ are gauge-equivalent} \Longleftrightarrow [\alpha] = [\alpha'] \in H^1_{DR}(M).$$

2. Let M be a differentiable manifold and $E \to M$ a vector bundle with connection ∇. For $A \in \Omega^1(End(E)$ the map defined by $\nabla + A$ is also a connection on E. Show, by computing in terms of a local frame, that the following holds

$$F^{\nabla + A} = F^\nabla - \frac{1}{2}[A, A] + d^\nabla A$$

3. Let $E \to D^n$ be a vector bundle.

a) Prove that E is trivial. (Hint: Consider D^n as the interior of the unit n-cube and do induction on n).

b) Suppose ∇ is a flat connection on E. Show that E admits a trivialization by parallel sections.

b) A connected manifold M is called *simply-connected* if every map from S^1 to M is homotopic to a constant map. Show that over simply-connected manifolds every vector bundle which admits a flat connection is trivial.

4. a) Consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$. Let ϕ denote the antipodal map $S^n \to S^n$, $p \mapsto -p$. Show that ϕ^* acts as $(-1)^{n+1}$ on $H^n_{DR}(S^n)$. (Hint: Consider the volume form ω, defined in a point p by

$$\omega_p = i_p \widetilde{\omega}, \quad p \in S^n \subset \mathbb{R}^{n+1},$$

where $\widetilde{\omega}$ is the standard volume form on $\mathbb{R}^{n+1}$.)

b) Suppose X is a vector field on S^n without zeroes. We can view the vector X_p at a point $p \in S^n$ in the standard way as an element of $\mathbb{R}^{n+1}$. Let $||X_p||$ denote its Euclidean norm. Show that the map

$$S^n \times [0,1] \to S^n, \ (p,t) \mapsto p\cos(\pi t) + \frac{X_p}{||X_p||}\sin(\pi t)$$

is well defined and a homotopy between the identity and the antipodal map.

c) Prove that every vector field on an even-dimensional sphere has a zero.

Please hand in your solutions on Friday, the 2nd of February, by 4pm.