Mathematisches Institut
Universität München

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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 12

1. a) Show that the tangent bundle of $S^3 \subset \mathbb{C}^2$ admits a global trivialization by vector fields E_1, E_2, E_3 such that $[E_i, E_{i+1}] = 2E_{i+2}$, where indices are mod 3.

b) Let $\theta_1, \theta_2, \theta_3$ be the basis of 1-forms dual to the E_i's. Define a connection ∇ on TS^3 by

$$\nabla E_i = \theta_{i+1} \otimes E_{i+2} - \theta_{i+2} \otimes E_{i+1},$$

(indices mod 3). Write down the matrix of connection 1-forms and compute the curvature matrix.

2. Let $E \to B$ be a vector bundle with connection ∇. Recall that the curvature $F^{\nabla} \in \Omega^2(End\, E)$ is defined via the composition $\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E)$. Show that

$$F^{\nabla}(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for all vector fields X, Y on B and all sections s of E.

Hint: Show that both sides of the equation are $C^\infty(B)$-linear in X, Y, s.

3. Let B be a connected differentiable manifold and $E \to B$ a vector bundle with connection ∇.

a) Show that a parallel section has a zero if and only if it vanishes identically.

b) Show that the set of parallel sections forms a finite dimensional vector space.

4. Let B be a differentiable manifold and $E \to B$ a vector bundle with connection ∇. Let s be a section of E, X a vector field on B and $p \in B$ a point. Suppose $c : I \to B$ is an integral curve of X with $c(0) = p$. Prove that

$$(\nabla_X s)(p) = \frac{d}{dt}(P_t^{-1}(s_{c(t)}))|_{t=0}$$

where $P_t : E_p \to E_{c(t)}$ is the parallel transport along c from 0 to t.

Please hand in your solutions on Friday, the 26th of January, by 4pm.