Mathematisches Institut
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Office hour: Wed 2-3 pm, Room 308 WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 11

1. Let $E \to M$ be a rank k vector bundle with connection ∇ and $\{s_i\}_{i=1}^k$, $\{s'_i\}_{i=1}^k$ local frames over an open set U. Define the matrix g by $s' = g \cdot s$, i.e. $s'_i(p) = \sum_{j=1}^k g_{ij}(p)s_j(p)$ in every $p \in U$. Show that the matrices of curvature 2-forms are related by $\Omega' = g \cdot \Omega \cdot g^{-1}$, i.e.

$$\Omega'_{ij} = \sum_{l,m=1}^{k} g_{il}\, \Omega_{lm}\, g^{mj}.$$

2. Let $E \to M$ be a rank k vector bundle and ∇ a connection. Consider an open covering $\{U_i\}$ of M such that for every i the restriction $E|_{U_i}$ is trivialized by a local frame. On each U_i we have the local curvature matrix Ω^i. We can consider polynomials in the entries of Ω^i. Show that $\mathrm{Tr}\Omega^i$ and $\det\Omega^i$, which are a priori only defined locally, patch together to give globally defined, closed differential forms $\mathrm{Tr}\Omega \in \Omega^2(M)$ and $\det\Omega \in \Omega^{2k}(M)$.

3. A connection on a vector bundle is called *flat* if it has vanishing curvature.

a) Show that every trivial vector bundle over a differentiable manifold admits a flat connection.

b) Find a vector bundle over $T^2 = S^1 \times S^1$ which is non-trivial but admits a flat connection.

4. Let $E \to M$ be a vector bundle with connection ∇ and curvature F^∇. Consider the vector bundle $End(E) \to M$.

a) Show that

$$(\overline{\nabla}_X L)(s) := \nabla_X(Ls) - L(\nabla_X s),$$

for $L \in \Gamma(End(E))$, $s \in \Gamma(E)$, $X \in \mathfrak{X}(M)$, defines a connection $\overline{\nabla}$ on $End(E)$.

b) Prove the following formula for the curvature $F^{\overline{\nabla}}$ of the connection $\overline{\nabla}$:

$$F^{\overline{\nabla}}(X,Y)L = [F^\nabla(X,Y), L].$$

(The term on the right hand side is the commutator of two endomorphisms of E.)

Please hand in your solutions on Friday, the 19th of January, by 4pm.