Mathematisches Institut
Universität München

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Office hour: Wed 2-3 pm, Room 308
WS 06/07

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 10

1. Let M and N be differentiable manifolds and $f, g : M \to N$ smoothly homotopic smooth maps. Prove that

$$f^* = g^* : H^k_{DR}(N) \to H^k_{DR}(M), \quad \text{for every } k \geq 0.$$

2. a) Let M be a compact oriented differentiable manifold without boundary and $p \in M$ a point. Show that the constant map $f : M \to M$, $x \mapsto p$, is not homotopic to the identity map.

b) Let $\phi_n : S^1 \to S^1$, $z \mapsto z^n$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the standard volume form ω on S^1. Show that $\phi_n^*\omega = n\omega$ and that the maps ϕ_n are pairwise non-homotopic.

3. Let ω be a volume form on D^2 and ϕ a compactly supported diffeomorphism of D^2 with $\phi^*\omega = \omega$. If $\omega = -d\lambda$, then prove

$$\int_{D^2} ((\phi^* df) \wedge \lambda + (\phi^*\lambda) \wedge df) = 0,$$

for all smooth functions f on D^2.

Hint: Use Stokes' Theorem and the fact that $\phi^*\lambda - \lambda = dg$ for a function g with compact support in D^2.

4. Let ω be an n-form on S^n with $\int_{S^n} \omega = 1$. For a smooth map $f : S^{2n-1} \to S^n$ define the *Hopf invariant*

$$H(f) = \int_{S^{2n-1}} \alpha \wedge d\alpha,$$

where $d\alpha = f^*\omega$ on S^{2n-1}. Prove the following:

a) $H(f)$ is well-defined independently of the choices of ω and α.

b) If $f_0, f_1 : S^{2n-1} \to S^n$ are smoothly homotopic, then $H(f_0) = H(f_1)$.

c) If n is odd, then $H(f) = 0$ for all f.

Please hand in your solutions on Friday, the 12th of January 2007, by 4pm.

Merry Christmas and a happy new year!