This is the set of notes for my seminar talk on 12.11.03. The goal is to prove the following theorem of Sinai.

Theorem 1. *Let M be a closed manifold and f: $M \longrightarrow M$ be an Anosov diffeomorphism. Then*
(i) Any diffeomorphism $f^{'}$ close enough to f in the C^1 topology is also Anosov.
(ii) The stable and unstable distributions of f are integrable.

Throughout the proof of this theorem we will use a (λ, μ)-definition of Anosov diffeomorphism which is a little different from the standard one, namely the inequalities are written as follows:

$$||Df^n w|| \leq \lambda^n ||w||\,, \;\; w \in E^s\,, n = 1, 2...,$$
$$||Df^n w|| \geq \mu^n ||w||\,, \;\; w \in E^u\,, n = 1, 2...,$$

for some constants λ and μ with $\lambda < 1 < \mu$. Note, that it is enough in this definition to write the inequalities for $n = 1$ only. The question of how the two definitions match is discussed later.

Proof of the Theorem. The most naive way to prove (i) would be to construct distributions $E^{s'}$ and $E^{u'}$ for $f^{'}$ and prove the hyperbolic inequalities for them, but this meets certain difficulties. Our approach is based on the concept of cones. The key idea is to work out an equivalent way of saying a diffeomorphism is Anosov. Namely, to replace the stable and unstable distributions with a family of cones. The concept of cones turns out to be more flexible than that of stable and unstable distributions, i.e. for one Anosov diffeomorphism many families of cones ensure the Anosov property and one family of cones ensure the Anosov property for many diffeomorphisms, in fact for an open set of them. This last observation is crucial for us.

Definition 1. Given an n dimensional Riemannian manifold M and a $(k, n-k)$ splitting $E^h \oplus E^v = TM$ (not necessarily continuous) of its tangent bundle, we define a family a cones $\mathcal{F}^\gamma = \{H_x^\gamma\}_{x \in M} \cup \{V_x^\gamma\}_{x \in M}$ as follows:

$$H_x^\gamma = \{\xi + \eta \in T_x M : \xi \in E_x^h, \eta \in E_x^v, ||\eta|| \leq \gamma ||\xi||\},$$

$$V_x^\gamma = \{\xi + \eta \in T_x M : \xi \in E_x^v, \eta \in E_x^h, ||\eta|| \leq \gamma ||\xi||\},$$

where γ is a positive number. We say that the family $\mathcal{F}^\gamma$ is generated by the splitting $E^h \oplus E^v = TM$.

Sometimes we talk about a family of cones without mentioning its generating splitting. Given a diffeomorphism f on a Riemannian manifold M we write a set of conditions which a family of cones $\mathcal{F}^\gamma$ may satisfy.

(1)
$$(Df)(H_x^\gamma \setminus \{0\}) \subset IntH_{f(x)}^\gamma\,,\ (Df^{-1})(V_x^\gamma \setminus \{0\}) \subset IntV_{f^{-1}(x)}^\gamma\,,\ x \in M,$$

(2)
$$||Df_x w|| \geq \mu' ||w||\,,\ w \in H_x^\gamma\,,\ x \in M,$$

(3)
$$||Df_x w|| \leq \lambda' ||w||\,,\ w \in V_x^\gamma\,,\ x \in M.$$

We say that a family of cones $\mathcal{F}^\gamma$ is f-hyperbolic if the conditions above hold true with some positive constants $\lambda' < 1 < \mu'$. Note that the notion of f-hyperbolicity behaves nicely if you vary f. More precisely we have the following

Remark 1. If a family $\mathcal{F}^\gamma$ is f-hyperbolic for some diffeomorphism f, then it is f'-hyperbolic for all diffeomorphisms f' sufficiently close to f in C^1 topology.

It is now natural to ask how hyperbolic families of cones are related to Anosov diffeomorphisms. For this assume we have fixed a reference metric for every Anosov diffeomorphism. Next, to each Anosov diffeomorphism f we assign a family of cones $\mathcal{F}_f^\gamma$ (and say the family is assigned) by saying that it is generated by the hyperbolic splitting of TM associated with f.

Proposition 1. *Let f be an Anosov diffeomorphism. Then its assigned family of cones $\mathcal{F}_f^\gamma$ is f-hyperbolic if γ is sufficiently small.*

The proof is nothing more than repeated application of the triangle inequality and we leave it for the reader. One may convince his(her)self by drawing sample cones in $\mathbb{R}^2$. To complete the picture we need a kind of converse.

Proposition 2. *Let f be any diffeomorphism. Assume there exists family of cones $\mathcal{F}^\gamma = \{H_x^\gamma\}_{x\in M} \cup \{V_x^\gamma\}_{x\in M}$ which is f-hyperbolic. Then f is Anosov.*

Propositions 1 and 2 tell us that for a diffeomorphism f to be Anosov is the same that to have an f-hyperbolic family of cones. This together with Remark 1 completes the proof of the first part of Theorem 1. To prove Proposition 2 we need an auxiliary

Lemma 1. *Let $p : E \longrightarrow M$ be a real vector bundle of rank n with a metric. Assume that for any point $x \in M$, the fiber $E_x = p^{-1}(x)$ carries the following structure: E_x^h and E_x^v are k and $n-k$ linear subspaces of dimensions k and $n-k$ respectively transverse to each other (the direct sum gives us the whole E_x); A_x^h and A_h^v are closed subsets of*

E_x containing E_x^h and E_x^v respectively. Let $\Phi : E \longrightarrow E$ be a bundle isomorphism with the following properties:

$$\Phi A_x^h \subset A_{f(x)}^h \, , \; \Phi^{-1} A_x^v \subset A_{f^{-1}(x)}^v , \tag{4}$$

$$||\Phi(u,v)|| \geq \mu' ||(u,v)|| \, , \; (u,v) \in A_x^h, x \in M, \tag{5}$$

$$||\Phi(u,v)|| \leq \lambda' ||(u,v)|| \, , \; (u,v) \in A_x^v, x \in M. \tag{6}$$

Writing an element from E as pair (u,v) corresponds to the splitting $E_x = E_h^h \oplus E_x^v$. Here λ' and μ' are some positive numbers with $\lambda' \leq \mu'$. Set $T_x^i = \Phi^i A_{\Phi^{-i}x}^h$, $R_x^i = \Phi^{-i} A_{\Phi^i x}^v$, $E_x^+ = \bigcap_{i=0}^\infty T_x^i$, $E_x^- = \bigcap_{i=0}^\infty R_x^i$. Then E_x^+ is a k dimensional linear space inside A_x^h and E_x^- is an $n-k$ dimensional linear space inside A_x^v.

Proof As it follows from (4), $T_x^{i+1} \subset T_x^i \subset T_x^0 = A_x^h$. This means that E_x^+ is the intersection of a nested sequence of closed sets. Moreover, E_x^+ possesses a kind of semi-invariance property with respect to Φ, namely $\Phi E_x^+ \subset E_{\Phi(x)}^+$. Next construction shows that the set E_x^+ contains a linear space. Namely, since $E_x^h \subset A_x^h$, (4) implies that $S_x^j = \Phi^j E_{\Phi^{-j}(x)}^h \subset T_x^j$. For each linear space S_x^j take an ordered orthonormal basis $\{e_j^l\}^{l=1,\dots k}$. By extracting successive subsequences we can without loss of generality assume that the sequences $\{e_j^l\}_{j=0,1\dots}$ converge for all l. Passing to the limit $j \to \infty$ gives the set $\{e^l\}^{l=1,\dots k}$ of k orthonormal vectors. Let a k dimensional linear space S_x be spanned by these vectors, i.e. $S_x = Span\{e^l\}^{l=1,\dots k}$. The construction is finished and we have to verify that $S_x \subset E_x^+$. It is enough to check the corresponding inclusion for the basis $\{e^l\}^{l=1,\dots k}$ of S_x. To do this fix l and a particular choice of the indexes i, j with $i < j$. This gives us that $e_j^l \in S_x^j \subset T_x^j \subset T_x^i$. If you let j go to infinity the closeness of T_x^i will give you that $e^l \in T_x^i$. Since this is true for all $i = 0, 1, \dots$, we have that $e^l \in E_x^+$. And this is, in turn, true for all l, and hence the inclusion $S_x \subset E_x^+$ is verified. So now we need to prove the converse inclusion. For this we remark that S_x is transverse to E_x^v. Indeed, existence of a nonzero vector w, common to both linear spaces S_x and E_x^v would violate inequalities (5) and (6) when applied to w. Because the spaces S_x and E_x^v have complementary dimensions any $(u,v) \in E_x^+$ can be written as a sum $(u,v) = (u,v') + (0,v'')$, where $(u,v') \in S_x$, in a unique way. If we let

$$(u_j, v_j) = \Phi^{-j}(u,v) \, , \; (u_j' v_j') = \Phi^{-j}(u,v') \, , \; (u_j'' v_j'') = \Phi^{-j}(0,v'')$$

then $(u,v) \in E^+_x$ implies that $(u_j, v_j) \in A^h_{\Phi^{-j}(x)}$, and hence by (5), we have $||(u_j, v_j)|| \leq (\mu^{'})^{-j}||(u, v^{'})||$. By the same token $||(u^{'}_j v^{'}_j)|| \leq (\mu^{'})^{-j}||(u, v^{'})||$. Similarly, by (6), we have $||v^{''}|| \leq (\lambda^{'})^j||(u^{''}_j v^{''}_j)||$. Altogether:

$$||v^{''}|| \leq (\lambda^{'})^j||(u^{''}_j v^{''}_j)|| \leq (\lambda^{'})^j||(u_j, v_j)||+||(u^{'}_j v^{'}_j)|| \leq (\frac{\lambda^{'}}{\mu^{'}})^j(||(u,v)||+||(u,v^{'})||).$$

For all $j \in \mathbb{N}$, whence $v^{''} = 0$ and $(u,v) \in S_x$. This proves the inclusion $E^+_x \subset S_x$, and hence the equality $E^+_x = S_x$. Q.E.D.
Note that this, for dimension reasons, immediately gives the improvement of semi-invariance to the true invariance of $\{E^+_x\}_{x \in M}$ with respect to Φ, namely $\Phi E^+_x = E^+_{\Phi(x)}$.
Proof of Proposition 2 Let the vector bundle $p : E \longrightarrow M$ be the tangent bundle TM of our Riemannian manifold M. For every $x \in M$ let the subspaces E^h_x and E^v_x be those from the generating splitting for the family $\mathcal{F}^\gamma$ and set $A^h_x = H^\gamma$, $A^v_x = V^\gamma$. Set $E^s_x = E^-_x$ and $E^u_x = E^+_x$. Finally, set $\Phi = Df$. Apply Lemma 1. Inequalities (2) and (3) guarantee that the f-invariant splitting $E^s_x \oplus E^u_x = TM$ is, actually, hyperbolic. Hyperbolicity implies that the splitting is continuous. This shows that f is Anosov and completes the proof of Proposition 2. Q.E.D.

Let now f be Anosov in the standard definition. Take a natural number n to be so big that $\lambda = a^{1/n}exp(-b) < 1$, $\mu = a^{1/n}exp(b) > 1$. Then f^n is Anosov in the (λ, μ)-definition. If now $f^{'}$ is a C^1 small perturbation of f, then $(f^{'})^n$ is a C^1 small perturbation of f^n, and hence, by what we have just proved, the diffeomorphism $(f^{'})^n$ is (λ, μ)-Anosov, but then it is easy to see that $f^{'}$ is Anosov in the standard definition. This completes the proof of the first part of Sinai's Theorem.

For (ii), given an Anosov diffeomorphism f, fix a reference Riemannian metric g and consider the set $\mathcal{P}$ of all k-plane fields which we make into a complete metric space by introducing a distance function d. Namely, to every k-plane field Q we assign a family of linear operators $\{Pr(Q_x)\}_{x \in M}$ by saying that $Pr(Q_x)$ is an orthoprojector on the plane Q_x with respect to the Euclidean structure induced by g. Then the distance d is defined as follows:

$$d(Q_1, Q_2) = sup_{x \in M}||(Pr(Q_{1x}) - Pr(Q_{2x}))||.$$

All the axioms of metric are easily verified. Now one would like to say 'The metric space $\mathcal{P}$ is compact and hence complete.', but compactness is usually far from being easy to prove in function spaces like $\mathcal{P}$, though the proof of completeness follows the standard pattern.

Namely, given a Cauchy sequence of k-plane fields, you (pointwise) obtain a Cauchy sequence of orthoprojectors. The space of bounded linear operators is complete in the operator norm and the limit of a sequence of orthoprojectors is itself an orthoprojector. This gives you the limit orthoprojector assigned to every point. Using the compactness of M you argue that this assignment is continuous and gives you a true (not just pointwise, but in the metric above) limit plane field. The diffeomorphism f induces an operator $f^{\star\star}$ on the space $\mathcal{P}$ in a natural way. To every k-plane field Q we assign the field $f^{\star\star}(Q) = \{f^{\star\star}(Q)_x\}_{x\in M}$, where $f^{\star\star}(Q)_x = Df_{f^{-1}(x)}Q_{f^{-1}(x)}$. The operator $f^{\star\star}$ is a contraction an a neighbourhood U of the unstable plane field (distribution) E^u of f. This is easy and follows the pattern of Proposition 1. Since E^u is a fixed point of the contraction $f^{\star\star}$ we have that $E^u = \lim_{n\to\infty} f^{\star\star n}Q$ uniformly with respect to $Q \in U$.

Integrability is a local property, so it is enough to show that E^u is integrable at a point. Given a point $x \in M$ consider its successive preimages $x_l = f^{-l}(x)$, $l = 0, 1, 2, ...$ Around every point x_l take a chart (U_l, ϕ_l). Consider a foliation χ_l of U_l by parallel k-planes (with respect to the map ϕ_l) such that the tangent space to C_l at the point x_l coincides with E^u_x. Let X_l be a distribution tangent to χ_l. If U_l is chosen small enough, then X_l will be close to E^u in U_l. Using partition of unity we can extend the local distribution X_l to a global, not necessarily integrable distribution GX_l close to E^u, i.e. $GX_l \in U$. Thus we know that $f^{\star\star n}(GX_l)$ converges to E^u when n goes to infinity, uniformly with respect to l. In particular, $\lim_{n\to\infty} C_n = E^u$, where $C_n = f^{\star\star n}(GX_n)$ (the diagonal trick). All the C_n's are integrable at x, namely C_n integrates to $f^n(\chi_n)$ in the neighbourhood $f^n(U_n)$ of the point x. Because integrability is a closed condition, E^u is also integrable at x. Since x was arbitrary, the unstable distribution E^u of the diffeomorphism f is integrable. Integrability of the stable distribution can be obtained by replacing f with f^{-1}.

Let now f be Anosov in the standard definition, then, as we have already mentioned f^n is (λ, μ)-Anosov for some n, and hence has integrable stable and instable distributions, but these coincide with those of f. This completes the proof of Theorem 1.