Structural Stability

Seminar on Anosov Systems (Prof. Kotschick)

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Structural stability of diffeomorphisms

In this and the next section the required definitions are given.

Def: M, N smooth manifolds. $f \in C^r(M)$ and $g \in C^r(N)$ are called *topologically conjugate* iff there exists a homeomorphism $h \in C^0(M, N)$ such that

$$f = h^{-1} \cdot g \cdot h$$

Def: M, N, f, g as before. g is called a *(topological) factor* of f iff there exists a surjective continuous map $h : M \to N$ such that

$$h \cdot f = g \cdot h$$

Def: $f \in C^r(M)$ is called C^r - *structurally stable* iff there exists a neighborhood U of f (in the $C^r(M)$ - topology) such that every map $g \in U$ is topologically conjugate to f.

Remark: Note that, in general, the homeomorphism $h = h(g)$ is not uniquely defined. But there is a canonical way to make the notion of stability stronger.

Def: $f \in C^r(M)$ is called C^r - *strongly structurally stable* if the conjugacy $h = h(g)$ can additionally be chosen such that $g_n \to f$ in $C^r(M)$ implies $h_n \to \mathrm{id}$ in the $C^0(M)$ - topology.

Structural stability of flows

As one expects there exist similar notions for flows. But there are also very big differences, mainly the relaxed condition of just considering flows up to orbit equivalence. It is also important to notice that the topology for the flow sets come from the corresponding vector fields, i.e., $\phi : \mathbb{R} \times M \to M$ is defined to be C^r iff X is a C^r - vector field on M.

Def: M, N as before. Two C^r - flows $\phi_t : M \to M, \psi_t : N \to N$ are called C^0 - *flow equivalent* iff there exists a homeomorphism $h : M \to N$ such that

$$\phi_t = h^{-1} \cdot \psi_t \cdot h$$

for all $t \in \mathbb{R}$.

Remark: This obviously corresponds to topological conjugacy in the time - discrete case, since the defining equation for the conjugacy also holds for iterates of f, g. But flow equivalence is, in general, too restrictive. If one is interested in the dynamical behaviour of flows, like the number of equilibria, one should at least allow time shifts.

Def: Let ϕ_t, ψ_t be flows on M. ψ_t is called a *time change* of ϕ_t iff for all $x \in M$ the orbits coincide, i.e.,

$$\{\phi_t(x) : t \in \mathbb{R}\} \; = \; \{\psi_t(x) : t \in \mathbb{R}\}$$

and the flow orientation is preserved.

Def: M, N as before. Two C^r - flows $\phi_t : M \to M, \psi_t : N \to N$ are called C^0 - *orbit equivalent* iff there exists a homeomorphism $h : M \to N$ such that

$$\nu_t = h^{-1} \cdot \psi_t \cdot h$$

is a time change of ϕ_t. This is equivalent to requiring that

$$h(\{\phi_t(x) : t \in \mathbb{R}\}) \; = \; \{\psi_t(x) : t \in \mathbb{R}\}$$

and h preserves the flow orientation.

Def: A C^r - flow ϕ_t on M is called C^r - *(strongly) structurally stable* iff there exists a neighborhood U of ϕ_t (w.r.t. the $\Gamma^r(M)$ - topology induced by the vector fields) such that each flow $\psi_t \in U$ is C^0 - orbit equivalent to ϕ_t ($(\psi_n)_t \to \phi_t$ in $\Gamma^r(M) \Rightarrow h_n \to \mathrm{id}$ in the C^0 - topology, resp).

Toy model: Flow on D^n

In this section we consider flows on the unit ball $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\} \subset \mathbb{R}^n$. In order to ensure that the flow-lines stay in D^n for all positive times, we impose the natural requirement $< X(x), x > < 0$ for all $x \in \partial D^n$ on all vector fields $X : D^n \to \mathbb{R}^n$.

The content of this section entirely consists of stating and proving the following theorem:

Theorem: Let ϕ_t be a C^1 - flow on D^n such that
- ϕ_t has exactly one equilibrium $0 \in D^n$, which is a sink,
- $\lim_{t\to\infty} \phi_t(x) = 0$ for all $x \in D^n$.

Then ϕ_t is C^1 - structurally stable.

The proof consists of three steps:

First step: There exists a neighborhood $U \subset \Gamma^1(M)$ of ϕ_t and a neighborhood V of $0 \in D^n$ such that each $\psi_t \in U$ has a unique equilibrium a in V, which is again a sink.

This observation relies on the following generalized version of the implicit function theorem:

Lemma: Let $W \subset \mathbb{R}^n$ be open, $X : W \to \mathbb{R}^n$ a C^1 - vector field and $x_0 \in W$. Assume that $DX(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exist open neighborhoods $U \subset \Gamma^1(M)$ of X and V of x_0 such that for all $Y \in U$ we have
- $Y|V$ is injective,
- $X(x_0) \in Y(V)$.

Remark: The above statement follows from the lemma since $DX(0)$ for a sink is negative definite and thus clearly invertible. It follows that one can choose a unique $a \in V$ such that $0 = X(0) = Y(a)$. Here U can be chosen so small such that $|a - 0| < \epsilon$ (for arbitrary small $\epsilon > 0$) for all $Y \in U$ and such that a has the same index for Y as 0 has for X.

Second step: All the positive flowlines of $\psi_t \in U$ end in the associated sink a (with U possibly smaller).

Lemma: There exists a number $r > 0$ and a possibly smaller neighborhood U of X (= flow of ϕ_t) such that for each $Y \in U$ we have that $a \in B_r := \{x \in D^n : |x| \leq r\}$, B_r is in the basin of a and positively invariant under the flow ψ_t of Y.
(*basin* of $a := \{x \in D^n : \lim_{t\to\infty} \psi_t(x) = a\}$)

Proof: Define $2\nu := \max\{< DX(0) \cdot v, v >: |v| = 1\} < 0$. Hence we obtain

$$< DX(0) \cdot x, x > \; \leq \; 2\nu \cdot |x|^2$$

so that we can find some $r > 0$ such that

$$< X(x), x > \;\leq\; \nu \cdot |x|^2$$

for all $x \in B_{2r}$. Now for $U \subset \Gamma^1(M)$ small enough, one can prove that

$$< Y(x), x - a > \;\leq\; \mu \cdot |x - a|^2 \;, \;\; \mu < 0$$

for all $x \in B_r$ and $Y \in U$. This is the point where one can make explicit use of the property of Y of being C^1 - close. We start with

$$< Y(x), x - a > \;=\; < X(x - a), x - a > \;+\; < Y(x) - X(x - a), x - a >$$

If U is small enough such that $a \in B_\epsilon(0)$ with $\epsilon < r$ the first addend can be bounded by $\nu |x - a|^2$ for all $x \in B_r$. Hence it suffices to bound the second appropriately. Using Cauchy-Schwarz we have

$$< Y(x) - X(x - a), x - a > \;\leq\; |Y(x) - X(x - a)| \cdot |x - a|$$

so that we just need to make use of the Lipschitz property of $Z(x) := Y(x) - X(x - a)$ (Z is C^1),

$$|Z(x) - 0| \;\leq\; \eta \cdot |x - a|$$

By choosing V small enough, we obtain $\eta < -\nu$ and thus $\mu := \nu + \eta$ does the job.

All the statements in the theorem now follow from the above inequality for Y. q.e.d.

So far we have shown that some small neighborhood B_r of the sink lies in the basin of a for all $Y \in U$. It remains to prove that this extends to all of D^n. But this can shown by simple continuity arguments, since all the equilibria lie in B_r.

For all $x \in \overline{D^n - B_r}$ there exists some $t \in \mathbb{R}$ such that $\phi_t(x) \in B_r$, since D^n is in the basin of ϕ_t. Since this set is compact, we find some maximal time t_0. It follows from continuity that $\psi_t(\overline{D^n - B_r}) \subset B_r$ for $t \geq t_0 + 1$ for Y in some (once again) possibly smaller $U \subset \Gamma^1(M)$. Therefore we have shown for all $Y \in U$ that

$$\lim_{t \to \infty} \psi_t(x) = a$$

for all $x \in D^n$ with ψ_t being the flow of Y.

Third step: Construction of the conjugacy homeomorphism $h \in C^0(D^n)$ for the orbits of ϕ_t and ψ_t. This proves the structural stability.

It needs to be shown that ϕ_t is C^0 - orbit equivalent to all flows ψ_t in the neighborhood U. The required homeomorphism can be determined explicitely by means of the flows ϕ_t and ψ_t. We will not only end up with orbit equivalence but even with flow equivalence:

Define maps $\Phi : [0,\infty) \times \partial D^n \to D^n - \{0\}$ and $\Psi : [0,\infty) \times \partial D^n \to D^n - \{a\}$ by $\Phi(t,x) = \phi_t(x)$ and $\Psi(t,x) = \psi_t(x)$. These maps are homeomorphisms. Note especially that for every point $D^n - \{\text{sink}\}$ there exists a unique point on the boundary, which 'flows' to it in a (unique) finite time.

With these we can construct the conjugacy $h : D^n \to D^n$ easily. Namely set $h(y) = (\Psi \cdot \Phi^{-1})(y)$ for $y \neq 0$ and $h(0) = a$.

Structural stability of Anosov diffeomorphisms

Well, what might the content of this last section ? Yes, you are right.

Theorem: Anosov diffeos on compact manifolds are (C^1 -) structurally stable.

Proof: We prove this by means of the implicit function theorem for Banach spaces. Since it is a purely local statement, it applies to Banach manifolds as well. However, we use here a version with relaxed conditions. To be more precise, our proof relies on the following lemma:

Lemma: Let E be a topological vector space, F a Banach space. Let $f : U \times V \to F$ ($U \subset E, V \subset F$) be C^1 with invertible Frechet derivative with respect to the second coordinate. Then there exists a mapping $g : U_1 \to V_1$ ($x_0 \in U_1 \subset U, y_0 \in V_1 \subset V$) such that $f(x,y) = f(x_0,y_0) \Longleftrightarrow y = g(x)$ for all $x \in U_1, y \in V_1$.

Proof of the lemma: see the paper *H. Gloeckner, Implicit Functions from Topological Vector Spaces to Banach Spaces, arxiv:math.GM/0303320.*

Therefore, let $D := \mathrm{Diff}(M)$, $C := C^0(M,M)$ and define a mapping $A : D \times C \times D \to C \times C$ with $A(g_1,h,g_2) := (h \cdot g_1, g_2 \cdot h)$. For the proof of the theorem we wish to find for chosen Anosov diffeomorphism f and some C^1 - close diffeo $g \in D$ two mappings $h, h' \in C$ such that $h \cdot f = g \cdot h$ and $h' \cdot g = f \cdot h'$. In view of the mapping A this corresponds to the search for some mapping u in some neighborhood of $(f, \mathrm{id}, f) \in D \times C \times D$ such that $h = u(g_1,g_2) \Longleftrightarrow A(g_1,h,g_2) \in \Delta$, where Δ denotes the diagonal in $C \times C$. With this map u we get $h = u(f,g)$ and $h' = u(g,f)$. In a last step it just remains to check $h \cdot h' = h' \cdot h = \mathrm{id}$, which can be guaranteed by shrinking the domains.

So, we need to check the requirements for applying the implicit function theorem to the mapping A. Therefore, we first need to check the local structure of the involved spaces C and D. The obstructions on D as being locally a topological vector space are obviously fulfilled. In order to turn C into a Banach manifold, let us construct a local model for C. First it is known that the space $\tilde{C} := C^1(M, M)$ lies dense in C with respect to the C^0 - topology and hence provides a skeleton. Its tangent space at a point $f \in \tilde{C}$ is given by the space of differentiable sections $\Gamma^1(f^*TM)$, which, of course, serves as a local model for $\tilde{C}$. The corresponding local model of C is now given by the space of *continuous* sections $\Gamma^0(f^*TM)$. It inherits the structure of a Banach space, if we define the norm for $X \in \Gamma^0(f^*TM)$ by $\|X\| := \sup_{p\in M} |X(p)|$, where $|\cdot|$ denotes some arbitrarily chosen Riemannian metric on M. Note that the Banach space topology is completely independent of the choice of the Riemannian metric, since the norm equivalence of $|\cdot|_1$ and $|\cdot|_2$ on $\mathbb{R}^n$ immediately carries over to an equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$ in the sense that $a\|\cdot\|_1 \leq \|\cdot\|_2 \leq b\|\cdot\|_1$ for X - independent $a, b > 0$.

The differentiability condition on A is easily seen, so remains to check that the Frechet derivative of A with respect to the C - coordinate is invertible in the point (f, id, f). Well, not quite, after a certain projection, which is due to the fact, that we just look for solutions of A in the diagonal Δ of $C\times C$.

So we have to prove that $p \cdot L$ is invertible, where the C - derivative $L = D_2A(f, \mathrm{id}, f)$ is given by

$$\begin{aligned} L : \Gamma^0(TM) &\longrightarrow \Gamma^0(f^*TM) \times \Gamma^0(f^*TM) \\ X &\mapsto (X \cdot f, Df \cdot X) \end{aligned}$$

and $p : \Gamma^0(f^*TM) \times \Gamma^0(f^*TM) \to [\Gamma^0(f^*TM) \times \Gamma^0(f^*TM)]/\sim$ denotes the projection onto the diagonal.

In order to do so, we first make use of the canonical isomorphism

$$\begin{aligned} S : [\Gamma^0(f^*TM) \times \Gamma^0(f^*TM)]/\sim &\longrightarrow \Gamma^0(TM) \\ (X, Y) &\mapsto X \cdot f^{-1} - Y \cdot f^{-1} \quad . \end{aligned}$$

Multiplying with $p \cdot L$ on the right gives us

$$(S \cdot p \cdot L)(X) = S(X \cdot f, Df \cdot X) = X - Df \cdot X \cdot f^{-1} = (\mathrm{id} - f_*)(X) \quad ,$$

so it suffices to prove that $\mathrm{id} - f_* : \Gamma^0(TM) \to \Gamma^0(TM)$ is an isomorphism.

This is the point where the Anosov property of f comes into play. By definition, there exists a splitting of the tangent bundle of M into stable and unstable distributions, $TM = E^s \times E^u$. Due to the fact that Df preserves E^s, E^u, the pushforward of f can be split into two mappings $f_0 := f_*|_{\Gamma^0(E^s)}$ and $f_\infty := f_*|_{\Gamma^0(E^u)}$. By the Anosov property there exist $C > 0$ and $0 < \lambda < 1$ such that for all $m \in \mathbb{N}$ we have $\|f_0^m\|, \|f_\infty^{-m}\| < C \cdot \lambda^m$. Therefore we can make use of the Neumann method for constructing inverses of bounded linear maps, since the occuring series expansions converge,

$$\begin{aligned} (f_0 - \mathrm{id})^{-1} &= -\sum_{m=0}^{\infty} f_0^m \\ (f_\infty^{-1} - \mathrm{id})^{-1} &= -\sum_{m=0}^{\infty} f_\infty^{-m} \quad , \end{aligned}$$

which proves that $f_0 - \mathrm{id}$ and $f_\infty - \mathrm{id} = -f_\infty(f_\infty^{-1} - \mathrm{id})$ are isomorphisms. This then also holds for $f_* - \mathrm{id}$ by the splitting property.

So we have shown the first part of the proof of the theorem. As already stated, it remains to show that we can find h, h' such that $h \cdot h' = h' \cdot h = \mathrm{id}$. This is done by simple shrinking of the domains U_1 and V_1.

Therefore, choose a smaller neighborhood $U_2 \subset U_1$ of id in C such that $h_1 \cdot h_2 \in U_1$ for $h_1, h_2 \in U_2$. Further adjust V_1 to the smaller neighborhood of id, i.e., choose a neighborhood $V_2 \subset V_1$ of f such that $u(g_1, g_2) \in U_2$ for $g_1, g_2 \in V_2$. Then, for any $g \in V_2$ we have that h and h' are in U_2 (f is in V_2 anyway) and therefore, by definition, $h \cdot h', h' \cdot h \in U_1$. Hence, $h \cdot h'$ and $h' \cdot h$ are the unique solutions of

$$\begin{aligned} (h \cdot h') \cdot g &= h \cdot f \cdot h' = g \cdot (h \cdot h') \\ (h' \cdot h) \cdot f &= h' \cdot g \cdot h = f \cdot (h' \cdot h) \end{aligned}$$

in U_1 and therefore equal to the identity map. This proves the theorem.