This is the set of notes for my seminar talk on 28.01.04.

Projectively Anosov flows are a certain generalization of Anosov flows. Weakening assumptions on a flow in particular results in possibly having compact leaves of stable and unstable foliations which is impossible for Anosov flows. If you have a compact leaf, then it must be a 2-torus. The first question one is wondering about here is whether it is possible or not to extend a given flow on the 2-torus T^2 to the larger manifold $T^2 \times (-\epsilon, \epsilon)$, such that the extension becomes a projectively Anosov flow.

We start with necessary notations and definitions. Let M denote a closed oriented 3-manifold and $\{\Phi_t\}_{t\in\mathbb{R}}$ a smooth (C^∞) flow on it. Let $S^1(TM/T\Phi)$ be the S^1-bundle over M, which is obtained by subtracting the zero section from the rank 2 vector bundle $(TM/T\Phi)$ and then identifying points which differ by a multiplication with a positive number.

Definition 1. A flow $\{\Phi_t\}$ is called a projectively Anosov (pA) flow if there are four continuous sections $\mathcal{E}^u_\pm$, $\mathcal{E}^s_\pm$ in $S^1(TM/T\Phi)$ which are invariant under the action of $\{\Phi_t\}$, such that any orbit which is not contained in $\mathcal{E}^u_\pm$ or $\mathcal{E}^s_\pm$ is attracted to $\mathcal{E}^u_\pm$ (respectively $\mathcal{E}^s_\pm$) when $t \to \infty$ (respectively $t \to -\infty$).

As it follows directly from the definition, for a pA flow $\mathcal{E}^u_+$ and $\mathcal{E}^u_-$ are antipodal to each other and similarly for $\mathcal{E}^s_\pm$. We remark that any Anosov flow is projectively Anosov. Indeed, having strong stable E^{ss} and unstable E^{uu} distributions, set $\mathcal{E}^s_+ = e^s$ and $\mathcal{E}^s_- = -e^s$, where e^s is a vector field which spans E^{ss}, viewed as a section of $S^1(TM/T\Phi)$. And analogously for $\mathcal{E}^u_\pm$. The converse is not always true, i.e not every projectively Anosov flow is Anosov. Although some concepts used for Anosov flows can be generalized to the case of projectively Anosov flows. For a pA flow one defines continuous plane fields $E^{u(s)}$ by

$$x \mapsto (\mathcal{E}^{u(s)}_+(x) \cup \mathcal{E}^{u(s)}_- \cup z(x)) + T_x\Phi,$$

where z denotes the zero section of TM and x runs through M. It is known that the plane fields E^u and E^s are (in general not uniquely) integrable. In the case integral submanifolds are uniquely determined we call them $\mathcal{F}^u$ and $\mathcal{F}^s$ respectively. One can get unique integrability assuming for example that the plane fields E^u and E^s are of class C^1. The following result is due to Noda.

Theorem 1. *A C^∞ flow on T^2 without Reeb components can be extended to a projectively Anosov flow on $T^2 \times (-\epsilon, \epsilon)$ so that invariant foliations are of class C^1 and T^2 is a compact leaf of the unstable foliation.*

To prove this theorem we will need one criterion of being projectively Anosov, which we now state.

Proposition 1. *Suppose Φ^t is a pA flow with C^1-smooth stable and unstable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. Then $\mathcal{F}^u$ and $\mathcal{F}^s$ can be defined by C^1 1-forms α_u and α_s such that*

$$< \alpha_u, \alpha_s >= \alpha_u \wedge d\alpha_s + \alpha_s \wedge d\alpha_u > 0. \tag{1}$$

Conversely, suppose C^1 forms α_s and α_u satisfy the inequality (1) and define the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$, then any non-vanishing vector field on $\mathcal{F}^u \cap \mathcal{F}^s$ defines a pA flow.

Proof of the theorem. Assume that the flow Φ on T^2 has no Reeb components. Then the 1-foliation $\mathcal{F}^s$ it defines on T^2 can be defined by the smooth 1-form

$$\alpha = dy + \psi(x,y)dx, \tag{2}$$

for some coordinate system (x,y) on T^2 and some smooth function ψ. Consider the manifold $T^2 \times (-\epsilon, \epsilon)$ and let the last factor carry the natural coordinate z. From now on the coordinates (x,y,z) are fixed. We let the foliation $\mathcal{F}^s$ on $T^2 \times (-\epsilon, \epsilon)$ be defined by the smooth 1-form which satisfies (2) on the whole $T^2 \times (-\epsilon, \epsilon)$. To simplify the notation this 1-form is again denoted by α. Let the smooth foliation $\mathcal{F}^u$ be defined by the smooth 1-form

$$\beta = Azdx + dz,$$

where the positive constant A is not fixed for the moment. Then it holds that

$$< \alpha, \beta >= (A - \psi_y) dx \wedge dy \wedge dz.$$

This is positive for sufficiently large A. The application of Proposition 1 with $\alpha_s = \alpha$, $\alpha_u = \beta$ completes the proof of the theorem.

Now we discuss several examples of Anosov projectively Anosov flows. Consider the simple Lie algebra $psl(2,\mathbb{R})$ and fix the basis (h,l,k) for it: $h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $l = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & -0 \end{pmatrix}$, $k = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & -0 \end{pmatrix}$. One checks by hand the following commutator relations: $[h,l] = -l$, $[h,k] = k$. Take a lattice (co-compact discrete subgroup) Γ in $\mathbb{R}^3$ viewed as the universal cover of $SL(2,\mathbb{R})$ and consider the quotient $M = \mathbb{R}^3/\Gamma$. Let X, e^s and e^u be the vector fields coming from the left invariant vector fields corresponding to the Lie algebra elements h, l and k respectively. Then the above commutator relations mean that X is an Anosov vector field and e^s and e^u span the (strongly) stable and the unstable distributions respectively.

Take abelian Lie groups $\mathbb{R}^2$ and $\mathbb{R}$ together with their lattices $\mathbb{Z}^2$ and $\mathbb{Z}$ respectively. Take a unimodular hyperbolic matrix $A \in SL(2, \mathbb{Z})$, $traceA > 2$, and consider the semi-direct products S_A of the Lie groups and Γ_A of their lattices defined by the action $exp(tlogA)$ of $t \in \mathbb{R}$ on $\mathbb{R}^2$. Consider the Lie algebra s_a associated to the Lie group S_A. It is easy to compute that the Lie algebra s_a is a semi-direct product of the Lie groups $\mathbb{R}^2$ and $\mathbb{R}$ defined by the action $\tau log(A)$ of $\tau \in \mathbb{R}$ on $\mathbb{R}^2$. By changing the basis of $\mathbb{R}^2$ according to the eigenvalue decomposition of A and changing the basis of $\mathbb{R}$ by scaling with the log of the greatest eigenvalue of A we see that the Lie algebra s_a is, independently of the choice of A, isomorphic to the solvable Lie algebra $solv$, the semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$, defined by the action $\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\tau \in \mathbb{R}$ on $\mathbb{R}^2$. If we take the basis (k, l, h) of $solv$, according to the above semi-direct product decomposition, it is easy to calculate that the following commutator relations hold: $[h, l] = -l$, $[h, k] = k$. Take the lattice Γ_a the Lie group S_A and consider the quotient $M = S_A/\Gamma_a$. Let X, e^s and e^u be the vector fields coming from the left invariant vector fields corresponding to the Lie algebra elements h, l and k respectively. Then the above commutator relations mean that X is an Anosov vector field and e^s and e^u span the (strongly) stable and the unstable distributions respectively.

This last example gives rise to an EpA flow, i.e. a projectively Anosov flow, which is not Anosov. It is well-known that the manifold M we have just constructed admits another description, namely it is a T^2-bundle over S^1 obtained as a suspension of the hyperbolic linear diffeomorphism A of T^2. And the vector field X is the canonical horizontal vector field. Because the map A has fixed points, the vector field X has closed orbits. Then the flow of X can be deformed in a small neighbourhood of the a closed orbit in such a way that it is contracting there (i.e. is no longer Anosov), but it contracts in one direction much faster than in the other, i.e. is a pA flow. Next example shows how to construct an EpA flow on any T^2-bundle over S^1, not necessarily with hyperbolic monodromy.

A pair $\Pi = (\xi_0, \eta_0)$ of contact plane fields is called pre-bi-contact structure (pbC) if there exists a common smooth non-singular Legendrian vector field X_Π. Its singular set Σ_Π is defined to be $x|\xi_{0x} = \eta_{0x}$, which easily turns out to be smooth tori, tangent to X_Π. A pbC structure with non-empty singular set is called essential pbC (EpbC for short) structure. Any EpbC structures can be modified into a bi-contact structure as follows. Let α_0 and β_0 be contact 1-forms which define (ξ_0) and (η_0) respectively. and γ be nonsingular 1-form such that $\gamma(X_\Pi) > 0$. Then it is easy to see that for sufficiently small $\epsilon > 0$ a new pair of contact 1-forms $(\alpha, \beta) = (\alpha_0, \beta_0 + \epsilon\gamma)$ defines a bi-contact structure. On T^3 with its standard coordination (x, y, z), a pair of contact 1-forms

$$(\alpha_0, \beta_0) = (cos(2\pi z)dx + sin(2\pi z)dy, cos(2\pi z)dx - sin(2\pi z)dy)$$

defines an EpbC structure with its singular set $\{z \in \frac{1}{4}\mathbb{Z}\}$, and the common Legendrian vector field is given by $\partial/\partial z$.