spring term 2022 Prof. D. Kotschick
Dr. J. Stelzig

Selected Topics in Complex Geometry II

Sheet 09

Exercise 1. Show that the Albanese map, defined in the lecture, is a holomorphic map.

Exercise 2. Let X be a compact Kähler manifold and α_X the associated Albanese map. Show that the pullback $\alpha_X^* : H^1(Alb(X),\mathbb{Z}) \to H^1(X,\mathbb{Z})$ is an isomorphism.

Exercise 3. Show by directly, i.e. without using the Theorem that compact Kähler manifolds are formal (or its proof), that projective spaces $\mathbb{C}\mathrm{P}^n$ and tori $\mathbb{C}^n/\Lambda$ are formal.

Exercise 4. Let

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z_i \in \mathbb{C} \right\}$$

and $\Lambda = H \cap GL_3(\mathbb{Z}[i])$. Show that the Iwasawa manifold $X := H/\Lambda$ is not formal, by exhibiting a nontrivial triple Massey product.

You may use without proof that the subcomplex of left-invariant forms computes the whole cohomology. I.e., writing $\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - z_1 dz_2$, the inclusion of the exterior algebra generated by the ω_i and their conjugates into the space of all forms

$$\Lambda(\omega_1, \omega_2, \omega_3, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) \subseteq \mathcal{A}(X)$$

is a quasi-isomorphism.

Hand-in: to Jonas Stelzig until Monday June 27th, 14:00.