spring term 2022 Prof. D. Kotschick
Dr. J. Stelzig

Selected Topics in Complex Geometry II

Sheet 10

Exercise 1. Let C be a compact curve of genus 2 and $p : D \to C$ a two-sheeted covering space of D. Let $E = \mathbb{C}/\Gamma$ an elliptic curve. Set

$$X := (D \times C)/\sigma$$

where sigma is an involution ($\sigma^2 = 1$) acting as the Deck transformation of p on the left and as multiplication by (-1) on the right. Show:

1. The action of σ is free (so X is a complex manifold).
2. The Hodge numbers of X are that of a complex 2-torus.
3. The image of the Albanese map of X is one-dimensional. In particular, X is not a 2-torus.

Exercise 2. Let $L \to E$ be a very ample line bundle over an elliptic curve $E = \mathbb{C}/\Gamma$. Let ϕ, ψ be two holomorphic sections of L that do not simultanously vanish. Set:

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad J_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad J_4 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show

1. The group Γ generated by the $J_k \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ for $k = 1, .., 4$ is a lattice in every fibre of $L \oplus L \to E$. In particular, each fibre of $\pi : M := (L \oplus L)/\Gamma \to E$ is a complex torus.
2. M is diffeomorphic to a real 6-torus.
3. The relative canonical bundle is $K_{M/E} = \pi^* L^{-2}$.

   [Remark: One can show that if M were Kähler, the relative canonical bundle would have to enjoy a positivity property contradicted by this. So M is a non-Kähler complex structure on a real 6-dimensional torus.]

Hand-in: to Jonas Stelzig until Monday June 27th, 14:00.