Riemannian geometry
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(1) In this course we shall assume a basic knowledge of smooth manifolds and smooth vector bundles.

(2) First we recall the notion of a connection on a vector bundle. Let $E \to B$ be a differentiable vector bundle of rank k over a smooth manifold B of dimension n.

Definition 1. A **connection** on E is an $\mathbb{R}$-linear map
$$\nabla \colon \Gamma(E) \longrightarrow \Omega^1(E)$$
satisfying the Leibniz rule
$$\nabla(fs) = df \otimes s + f\nabla(s)$$
for all $f \in C^\infty(B)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*B \otimes E)$ is the space of 1-forms on B with values in E. One can evaluate the 1-form on a vector field X to obtain
$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E) \ .$$
The Leibniz rule is then equivalent to
$$\nabla_X(fs) = (L_X f)s + f\nabla_X(s)$$
for all $X \in \mathcal{X}(B)$.

(3) A trivialization always induces a connection, called the product connection, or trivial connection. In particular, trivial bundles have connections.

(4) We prove the following fundamental properties of connections:
- A connection ∇ does not increase the support of sections, i. e. if $s \in \Gamma(E)$ vanishes on some open set $U \subset M$, then so does $\nabla(s)$.
- The value of $\nabla(s)$ at a point $p \in B$ depends only on the restriction of s to an orbitrarily small open neighbourhood of p. (In other words, ∇ is a differential operator, and $\nabla(s)(p)$ depends only on the germ of s at p.)
- If ∇_1 and ∇_2 are connections, then so is $t\nabla_1 + (1-t)\nabla_2$ for all $t \in \mathbb{R}$.
- If ∇_1 and ∇_2 are connections, then $\nabla_1 - \nabla_2 \in \Omega^1(End(E)) = \Gamma(T^*B \otimes E^* \otimes E)$.

(5) Using these properties and a partition of unity subordinate to a covering of M by open sets over which the restriction of E is trivial, we prove:

Proposition 2. *Every vector bundle E admits connections. The space of all connections on E is an affine space for the space $\Omega^1(End(E))$ of 1-forms on M with values in $End(E)$.*

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(6) A **metric** of signature (p, q) with $p + q = k = rk(E)$ on a vector bundle E is a fiberwise non-degenerate symmetric bilinear form of signature (p, q) on the fibers, which varies smoothly with the basepoint of the fiber.

(7) Definite metrics (with $q = 0$ or $q = 0$) always exist (use partitions of unity), but indefinite ones may or may not exist.

(8) A connection ∇ is said to be **compatible** with a given metric g if

$$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t)$$

for all $s, t \in \Gamma(E)$.

(9) The product connection with respect to a local trivialization given by an orthonormal frame is compatible with the metric.

(10) Using again a partition of unity subordinate to a covering of B by open sets over which the restriction of E is trivialized by orthonormal frames, we prove:

Proposition 3. *Every vector bundle E with a metric g admits compatible connections. The space of all compatible connections is an affine space for the space $\Omega^1(SEnd(E))$ of 1-forms on B with values in $SEnd(E)$, the bundle of endomorphisms which are skew-symmetric with respect to g.*

(11) Fix a local frame $s_1, \ldots, s_k$ for the restriction of E to a trivializing open set in M. This choice determines a product connection ∇_0 defined by the requirement $\nabla_0(s_i) = 0$ for all i. Every other connection ∇ differs from ∇_0 by the addition of a 1-form with values in $End(E)$. However, the given trivialization of E induces a trivialization of $End(E)$, and so a 1-form with values in $End(E)$ is nothing but a $k \times k$ matrix of ordinary 1-forms. Thus ∇ can be expressed by the matrix $\omega = (\omega_{ij})$ of 1-forms given by

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

(12) If E is equipped with a metric of signature (p, q) and the frame is orthonormal in the sense that $g(s_i, s_j)$ vanishes for $i \neq j$, is $+1$ for $i = j \leq p$, and is -1 for $i = j \geq p + 1$, then for a g-compatible connection the local connection matrix $\omega = (\omega_{ij})$ has the following skew-symmetry property:

$$\omega_{ij} g(s_j, s_j) = -\omega_{ji} g(s_i, s_i) \ .$$

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(13) Next we extend the differential operator given by a connection ∇ to bundle-valued forms of higher degree.

Lemma 4. *For every connection ∇ on $E \to B$ there is a unique $\mathbb{R}$-linear map*

$$\bar{\nabla} \colon \Omega^l(E) \longrightarrow \Omega^{l+1}(E)$$

which satisfies

$$\bar{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^l \omega \wedge \nabla(s) \qquad (1)$$

for all $\omega \in \Omega^l(M)$ and $s \in \Gamma(E)$. Moreover, this operator satisfies

$$\bar{\nabla}(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f \bar{\nabla}(\omega \otimes s) \qquad (2)$$

for all smooth functions f on B.

(14) Consider the operator $\bar{\nabla} \circ \nabla \colon \Omega^0(E) \longrightarrow \Omega^2(E)$ associated with a connection ∇ on E. It turns out that this is linear over $C^\infty(B)$, and is therefore given by an element $F^\nabla \in \Omega^2(End(E))$. This is called the **curvature** of ∇.

(15) The curvature F^∇ can be evaluated on pairs of vector fields $X, Y \in \mathcal{X}(B)$ to obtain an endomorphism, which is then applied to a section s of E. We have the following identity, whose proof we left as an exercise:

$$F^\nabla(X,Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \ .$$

Therefore, for commuting vector fields X and Y, the curvature measures the deviation from commutativity for the covariant derivatives ∇_X and ∇_Y.

(16) Fix a local frame $s_1, \ldots, s_k$ for the restriction of E to a trivialising open set in M. We saw already that a connection ∇ can be expressed by the matrix $\omega = (\omega_{ij})$ of 1-forms given by

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

From the definition of the curvature we calculate

$$F^\nabla(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} \ .$$

We can write this briefly as $\Omega = d\omega - \omega \wedge \omega$, where the wedge product on the right-hand-side includes matrix multiplication, and is therefore not necessarily trivial unless $k = 1$.

(17) Similarly we compute $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$. This is the **Bianchi identity**.

(18) As an example we consider vector bundles of small rank equipped with connections. If the rank is $= 1$, then the curvature is given by a single 2-form $\Omega_{11} = d\omega_{11}$. This is locally exact, and therefore closed.

If ∇ is compatible with a metric and the trivializing section is of unit length in this metric, then the skew-symmetry of the connection matrix means that ω_{11} vanishes, and therefore the curvature vanishes as well. As every bundle admits a metric and a compatible connection, we conclude that all rank one bundles admit connections with zero curvature.

If the rank is $= 2$ and the connection is compatible with a metric, then again the connection and curvature matrices with respect to a local orthonormal frame are skew-symmetric, and therefore are determined by just one entry, ω_{12} respectively Ω_{12}. Again we have that $\Omega_{12} = d\omega_{12}$, so the curvature form is closed.

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(19) We now discuss the **Euler class** of oriented rank 2 bundles $\pi \colon E \longrightarrow B$. We choose a positive definite metric and a compatible connection. With respect to an oriented orthonormal frame the curvature is determined by the closed 2-form Ω_{12}. This closed form is the same for all oriented orthonormal frames, and is therefore defined globally, even if the orthonormal frame is only defined locally. Therefore, we can define the Euler class by the formula

$$e(E) = -\frac{1}{2\pi}[\Omega_{12}] \in H^2_{dR}(B) \ .$$

(20) Changing the orientation of E changes the sign of its Euler class, since it interchanges Ω_{12} and $\Omega_{21} = -\Omega_{12}$.

(21) The Euler class does not depend on the metric connection ∇. Furthermore, it does not depend on the metric either, and is therefore a topological invariant of vector bundles.

(22) If E admits a nowhere vanishing section, then $e(E) = 0$. In the course of the proof we showed that an oriented rank 2 bundle is trivial if and only if it has a nowhere vanishing section.

(23) If an oriented rank 2 bundle admits a metric of signature $(1, 1)$, then its Euler class vanishes.

(24) If Σ is an oriented surface, then $T\Sigma$ is an oriented rank 2 bundle. One of the significant properties of the Euler class is that if Σ is compact without boundary then

$$-\frac{1}{2\pi}\int_{\Sigma}\Omega_{12} = \chi(\Sigma) = 2 - 2g(\Sigma)$$

is the Euler characteristic of Σ, where $g(\Sigma)$ is the genus of the surface. This is the **Gauss–Bonnet theorem**, which we do not prove today.

(25) On a smooth manifold M we now consider connections ∇ on the tangent bundle $TM \to M$. These are sometimes called **affine connections**. In this case the variables X and s in $\nabla_X s$ are on equal footing, as they are both sections of the tangent bundle. This leads to possible symmetries which make no sense in the more general setting of arbitrary vector bundles.

(26) The **torsion** of a connection ∇ on TM is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \mathcal{X}(M)$.

Lemma 5. *The torsion defines a skew-symmetric map*

$$T\colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

that is bilinear over $C^\infty(M)$.

A connection ∇ is called **symmetric** if it is torsion-free, i. e. if T vanishes identically[1].

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(27) We now prove the following:

Proposition 6. *On any pseudo-Riemannian manifold (M, g) there is a unique affine connection ∇ that is compatible with g and has a given two-form T with values in TM as its torsion.*

(28) As a special case of Proposition 6 we note the following:

Corollary 7. *On every pseudo-Riemannian manifold (M, g) there is a unique torsion-free metric affine connection.*

This connection is called the **Levi-Civita connection** of (M, g).

[1]Note that requiring the naive symmetry $\nabla_X Y = \nabla_Y X$ for all X and Y leads to a contradiction.

(29) *First proof of Proposition 6.* Let us suppose that there are affine connections compatible with g, and pick one, called ∇_0, as a reference. Then the space of all affine connections compatible with g is identified with

$$\Omega^1(SEnd(TM)) = \Gamma(T^*M \otimes SEnd(TM)) \subset \Gamma(T^*M \otimes T^*M \otimes TM) .$$

The torsion is a map from this space to

$$\Omega^2(TM) = \Gamma(\Lambda^2 T^*M \otimes TM)$$

sending A to T^{∇_0} plus the skew-symmetrization of A, see (6) above.

Assume that $\nabla_0 + A_1$ and $\nabla_0 + A_2$ have the same torsion. Then we can calculate that $A_1 = A_2$. Thus the torsion map is injective. But, at every point, it is an affine map between the fibers of two vector bundles of the same rank. Thus, if it is injective, it is also surjective. □

(30) We want to give an alternative proof for Proposition 6, which produces a formula for the desired connection and therefore does not assume that connections compatible with g exist. Instead of assuming this, we prove it as a consequence of the following argument.

Second proof of Proposition 6. We first prove uniqueness. So let ∇ be an affine connection compatible with g having torsion tensor T. Then using compatibility with g and the formula for the torsion alternately three times, we find that ∇ must be determined by the following formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}(g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) - g(X,T(Y,Z)) + \\ + g(Y,T(Z,X)) + g(Z,T(X,Y)) + L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y))) .$$

To prove existence, we use this equation as a definition for $\nabla_X Y$. This defines a connection, and one can check that it is both compatible with g and has $T^\nabla = T$. □

(31) We consider the expression of an affine connection in a local coordinate system $(x_1, \dots, x_n)$ on M. We write ∂_i for the coordinate vector fields $\frac{\partial}{\partial x_i}$, and use the local frame $\partial_1, \dots, \partial_n$. Then

$$\nabla \partial_i = \sum_{j=1}^{n} \omega_{ij} \otimes \partial_j ,$$

with the connection matrix

$$\omega_{ij} = \sum_{k=1}^{n} \omega_{ij}^k dx_k .$$

This gives

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \omega_{jk}^i \partial_k ,$$

which is usually written as

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$$

in classical notation. Therefore, we define the **Christoffel symbols** of the affine connection ∇ with respect to the coordinate system $(x_1, \dots, x_n)$ to be $\Gamma_{ij}^k = \omega_{jk}^i$.

Returning to the definition of torsion, we see that

$$T^{\nabla}(\partial_i, \partial_j) = \sum_{k=1}^{n}(\omega^i_{jk} - \omega^j_{ik})\partial_k = \sum_{k=1}^{n}(\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k \ .$$

As the torsion is linear over the smooth functions, we obtain the following:

Lemma 8. *An affine connection ∇ is torsion-free if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$ for any local coordinate system.*

Thus symmetry of an affine connection really refers to a symmetry of the Christoffel symbols expressing this connection in local coordinates.

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(32) Continuing the discussion from the previous lecture, the Levi-Civita connection of a pseudo-Riemannian manifold (M, g) has the following local expression in a local coordinate system $(x_1, \ldots, x_n)$:

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{n}(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ki})g^{lk} \ ,$$

where $g_{ij} = g(\partial_i, \partial_j)$, and (g^{lk}) is the matrix inverse to $g_{ij})$.

(33) By Corollary 7 every pseudo-Riemannian manifold (M, g) has a unique torsion-free affine connection compatible with the metric. This is called the **Levi-Civita connection** of (M, g), or of g.

(34) The **Riemann curvature tensor** R of a pseudo-Riemannian manifold (M, g) is defined using its Levi-Cività connection:

$$R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$
$$(X, Y, Z) \longmapsto R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ .$$

Recall that for an arbitrary connection ∇ on a vector bundle E, the curvature F^{∇} is a two-form on the base with values in $End(E)$. Here $R = F^{\nabla}$ for the Levi-Civita connection ∇. The notation $R(X, Y)Z$ means that the two-form is evaluated on X and Y, and the resulting endomorphism is then applied to Z.

(35) The most important properties of the Riemann curvature are the following:
  (i) it is trilinear over $C^{\infty}(M)$,
  (ii) it is skew-symmetric in the first two arguments,
  (iii) it satisfies the **first Bianchi identity**: $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ for all X, Y and Z,
  (iv) $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$ for all X, Y, Z and W, and
  (v) $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$ for all X, Y, Z and W.

(36) Let V be a real vector space with a non-degenerate symmetric bilinear form g (e.g. a tangent space of a pseudo-Riemannian manifold). A subspace $U \subset V$ is called **non-degenerate** if the restriction $g|_U$ is non-degenerate. A non-zero vector $X \in V$ is called **null** if it spans a degenerate subspace.

(37) For a two-dimensional subspace $\sigma \subset V$ non-degeneracy is equivalent to the non-vanishing of $Q(X, Y) = g(X, X)g(Y, Y) - (g(X, Y))^2$, for any basis X and Y of σ.

(38) Now let (M, g) be a pseudo-Riemannian manifold with Riemann tensor R.

Definition 9. For a non-degenerate tangent two-plane σ we define the **sectional curvature** to be

$$K(\sigma) = \frac{g(R(X,Y)Y,X)}{Q(X,Y)} \ .$$

This depends only on g and σ, and not on the basis chosen for σ.

(39) By definition, the sectional curvature of (M,g) is determined by its Riemann tensor. However, the converse is also true, by the following result we will not prove here.

Proposition 10. *The collection of sectional curvatures for all non-degenerate two-planes $\sigma \subset TM$ determines the Riemann curvature tensor R of (M,g).*

The proof uses the following:

Lemma 11. *Every pair of vectors X and Y can be approximated arbitrarily closely by X' and Y' such that the span of X' and Y' is a non-degenerate two-plane.*

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(40) We now work out the following special case of Proposition 10, characterizing spaces with constant sectional curvature:

Proposition 12. *A pseudo-Riemannian manifold (M,g) has sectional curvature equal to a fixed real number $K_0 \in \mathbb{R}$ for all non-degenerate two-planes $\sigma \subset TM$ if and only if the following identity holds for all X, Y, Z and $T \in \mathcal{X}(M)$:*

$$g(R(X,Y)Z,T) = -K_0(g(X,Z)g(Y,T) - g(Y,Z)g(X,T)) \ .$$

(41) For a pseudo-Riemannian manifold with Riemann tensor R we define the **Ricci tensor** to be

$$P(X,Y) = tr(Z \longmapsto R(Z,X)Y) \ ,$$

for all X and $Y \in \mathcal{X}(M)$. Here $tr(A)$ denotes the trace, i. e. the sum of the diagonal entries of any matrix representing the linear map A.

Definition 13. The **Ricci curvature** of an n-dimensional pseudo-Riemannian manifold (M,g) in the direction of $X \in \mathcal{X}(M)$ is

$$Ric(X) = \frac{1}{n-1} P(X,X) \ .$$

Clearly the Ricci curvature determines the Ricci tensor by polarization.

(42) We want to work out concrete formulae for the Ricci tensor and the Ricci curvature in terms of sectional curvatures. For this purpose we use a local orthonormal frame $X_1, \ldots, X_n$. Here orthonormal means that the X_i are pairwise orthogonal with respect to g, and that $g(X_i,X_i) = \pm 1$ for all i. (For a Riemannian manifold all these scalar products are $+1$.) With respect to such a local frame the trace of an endomorphism A of TM can be written as

$$tr(A) = \sum_{i=1}^{n} (g(AX_i,X_i) \cdot g(X_i,X_i)) \ .$$

Thus, we have

$$P(X,Y) = -\sum_{i=1}^{n} \left(g(R(X,X_i)Y,X_i) \cdot g(X_i,X_i)\right) \ ,$$

and

$$\begin{aligned} Ric(X) = \frac{1}{n-1}P(X,X) &= -\frac{1}{n-1}\sum_{i=1}^{n} \left(g(R(X,X_i)X,X_i) \cdot g(X_i,X_i)\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n} \left(g(R(X,X_i)X_i,X) \cdot g(X_i,X_i)\right) \ . \end{aligned}$$

If X is not a null vector, we may assume that

$$X_1 = \frac{1}{\sqrt{|g(X,X)|}}X \ ,$$

so that the above formula becomes

$$\begin{aligned} Ric(X) &= \frac{1}{n-1}\sum_{i=1}^{n} \left(g(R(X,X_i)X_i,X) \cdot g(X_i,X_i)\right) \\ &= |g(X,X)| \cdot \frac{1}{n-1}\sum_{i=2}^{n} \left(g(R(X_1,X_i)X_i,X_1) \cdot g(X_i,X_i)\right) \\ &= |g(X,X)| \cdot \frac{1}{n-1}\sum_{i=2}^{n} \left(K(Span\{X_1,X_i\}) \cdot g(X_1,X_1) \cdot g(X_i,X_i)^2\right) \\ &= g(X,X) \cdot \frac{1}{n-1}\sum_{i=2}^{n} K(Span\{X_1,X_i\}) \ . \end{aligned}$$

The last line shows that the Ricci curvature is essentially an average of sectional curvatures.

(43) We now want to define a scalar measure of curvature s by taking the trace of the Ricci tensor with respect to the metric g. Because g is non-degenerate, there exists a unique $A \in \Gamma(End(TM))$ such that

$$P(X,Y) = g(AX,Y)$$

holds for all vector fields X and Y.

Definition 14. The **scalar curvature** of a pseudo-Riemannian manifold (M,g) is the function s$\in C^\infty(M)$ defined by $s = tr(A)$.

Using a local orthonormal frame as before, and substituting from the above formula expressing the Ricci curvature in terms of sectional curvatures, we have the following:

$$\begin{aligned} s = tr(A) &= \sum_{i=1}^{n} (g(AX_i, X_i) \cdot g(X_i, X_i)) \\ &= \sum_{i=1}^{n} (P(X_i, X_i) \cdot g(X_i, X_i)) \\ &= (n-1) \sum_{i=1}^{n} (Ric(X_i) \cdot g(X_i, X_i)) \\ &= (n-1) \sum_{i=1}^{n} \left(\left(g(X_i, X_i) \cdot \frac{1}{n-1} \sum_{j \neq i} K(Span\{X_i, X_j\}) \right) \cdot g(X_i, X_i) \right) \\ &= \sum_{i \neq j} K(Span\{X_i, X_j\}) \\ &= 2 \sum_{i<j} K(Span\{X_i, X_j\}) \; . \end{aligned}$$

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(44) We now consider the following situation: (N, g) is a pseudo-Riemannian manifold, and $M \subset N$ is a smooth hypersurface, for which we assume that $T_pM \subset T_pN$ is a non-degenerate subspace with respect to g for all $p \in M$. This assumption ensures that $T_pN = T_pM \oplus (T_pM)^\perp$ for all $p \in M$, where $(T_pM)^\perp$ is the orthogonal space with respect to g. (This is not just a direct sum, but also an orthogonal sum with respect to g.) As M is a hypersurface it follows that $(T_pM)^\perp$ is one-dimensional, spanned by a vector n. By non-degeneracy this vector cannot be null, so that by scaling we may assume $g(n, n) = \pm 1$.

The non-degeneracy of the subspace $T_pM \subset T_pN$ for all $p \in M$ means that the restriction of g to TM is a pseudo-Riemannian metric h on M. The Levi-Civita connection ∇^M of (M, h) can be obtained from the Levi-Civita connection ∇ of (N, g) as follows:

$$\nabla^M_X Y = \pi \nabla_{\tilde{X}} \tilde{Y} \; ,$$

where $\tilde{X}$ and $\tilde{Y}$ are local extensions of $X, Y \in \mathcal{X}(M)$ to N and $\pi \colon TN|_M \to TM$ is the projection with kernel $(T_pM)^\perp$. As this is independent of the choices of local extensions for X and Y, we will not specify the extensions in later formulae.

(45) Suppose we have chosen a normal vector field n for $M \subset N$, normalized so that $g(n, n) = \pm 1$.

Definition 15. The **Weingarten map** at $p \in M$ is the linear map

$$\begin{aligned} L \colon T_pM &\longrightarrow T_pM \\ X &\longmapsto \nabla_X n \; . \end{aligned}$$

Again we need to choose a local extension of n in a neighborhood of p, but the result we get is independent of the choice of extension. It follows from the constancy of $g(n, n)$

that $\nabla_X n$ is orthogonal to n and therefore contained in T_pM. Note that the normalization determines n uniquely up to sign. If we replace n by $-n$, then we obtain $-L$ instead of L.

The Weingaten map has the following easily proved property, see [2], Section 10.3.

Lemma 16. *The Weingarten map is symmetric with respect to the metric, i. e.* $h(L(X), Y) = (X, L(Y))$ *for all* $X, Y \in T_pM$.

(46) For $X, Y, Z \in T_pM$ we can relate the value of $R(X, Y)Z$, the curvature of (N, g), and of $R^M(X, Y)Z$, the curvature of (M, h), through the Weingarten map. In the special case that the Levi-Civita connection ∇ of (N, g) is flat we obtain:

Theorem 17 (Gauss Equation)**.** *If the pseudo-Riemannian manifold* (N, g) *is flat, and* $M \subset N$ *is a hypersurface for which the restriction* h *of* g *to* M *is also pseudo-Riemannian, then the curvature of the Levi-Cività connection of* (M, h) *is given by*

$$R^M(X, Y)Z = g(n, n)\left(h(L(Y), Z)L(X) - h(L(X), Z)L(Y)\right) ,$$

where n *is a normal vector to* M *normalized so that* $g(n, n) = \pm 1$.

Notice that this formula is unchanged if we replace n by $-n$.

(47) As an example for the above situation we take $N = \mathbb{R}^{n+1}$, with g given by the standard positive definite scalar product. Note that because g is positive definite, all subspaces are positive definite as well, and are therefore non-degenerate.

For the hypersurface M we take the unit sphere $S^n \subset \mathbb{R}^{n+1}$. As normal vector field n we take the outer unit normal to the sphere, which at $p \in S^n$ is just p itself. Now we can calculate that in this case the Weingarten map L is the identity, so that the curvature tensor of the unit sphere endowed with the induced Riemannian metric h is given by

$$R^{S^n}(X, Y)Z = h(Y, Z)X - h(X, Z)Y .$$

Comparing this with Proposition 12 we see that (S^n, h) has constant sectional curvature equal to $+1$.

References

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