Smoothed Particle Hydrodynamics

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Smoothed Particle Hydrodynamics

- numerical method to simulate fluids (liquids, gases, plasmas)
- idea: represent fluid by moving particles
- first used in astrophysics
- increasingly used in CGI for block-buster movies
- upcoming technology for next-generation computer games

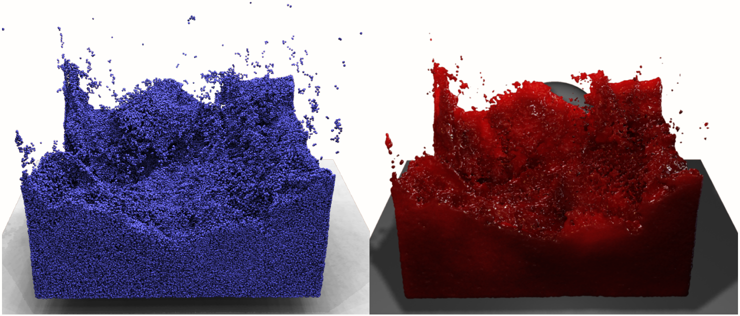

Figure : 1 Million particles, rendered in Maya, by Frank Zimmer

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4. Note that $A(r) = (A * \delta)(r) = \int A(r')\delta(r - r')\mathrm{d}r'$
5. Approximate $\delta(r - r')$ by $W_h(|r - r'|)$, where
    - $\int W_h(|r - r'|)\mathrm{d}r' = 1$
    - $W_h \overset{*}{\rightharpoonup} \delta$ for $h \to 0$
    - $W_h \in C_0^\infty$
    - supp $W_h \subset [0, \kappa h]$, $\kappa > 0$

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$$A_i = \sum_{j=1}^{N} m_j \frac{A_j}{\rho_j} W_h(|r_i - r_j|)$$

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- Gradient:

$$(\nabla A_S)(r) = \sum_{j=1}^{N} m_j \frac{A_j}{\rho_j} (\nabla W_h)(|r - r_j|) \frac{r - r_j}{|r - r_j|}$$

- Laplacian:

$$(\Delta A_S)(r) = \sum_{j=1}^{N} m_j \frac{A_j}{\rho_j} (\Delta W_h)(|r - r_j|)$$

Spatial derivatives

- SPH Field approximation

$$A_i = \sum_{j=1}^{N} m_j \frac{A_j}{\rho_j} W_h(|r_i - r_j|)$$

- SPH Gradient approximation:

$$(\nabla A)_i = \sum_{j=1}^{N} m_j \frac{A_j}{\rho_j} (\nabla W_h)(|r_i - r_j|) \frac{r_i - r_j}{|r_i - r_j|}$$

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Practical considerations

- Smoothing length h proportional to average particle diameter:

$$h \sim \frac{1}{\langle \rho \rangle^{\frac{1}{d}}}, \text{ where } \langle \rho \rangle := \frac{1}{n} \sum_{i=1}^{N} \rho_i$$

- Different kernels suitable for different charge densities.
- Kernels not C^∞ due to performance considerations (Splines!).
- Golden rules of SPH (Monaghan):
  - To find physical interpretation it's always best to assume kernel is Gaussian.
  - Rewrite formulas with mass density inside operators, by making use of

$$\nabla A = \frac{\nabla(A\psi)}{\psi} - \frac{A(\nabla\psi)}{\psi}$$

for positive smooth ψ.

Equations of motion

- Navier-Stokes equation:

$$\rho\left(\partial_t v + v \cdot \nabla v\right) = \rho g - \nabla p + \mu \Delta v, \text{ where}$$

  - v velocity
  - g gravity
  - p pressure
  - μ viscosity

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will be trivially satisfied: each particle has constant mass and particles are neither created nor destroyed.

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- Consider the total derivate of $v(r, t)$ with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t} v(r, t) = (\partial_t v)(r, t) + [\dot{r}(t)] \cdot (\nabla v)(r, t)$$

It depends on $\dot{r}$, where $r(t)$ is a *chosen* path in space.

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- Velocity v_i of particle moving with the fluid, i.e. $\dot{r}_i = v_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_i = \partial_t v_i + v_i \cdot \nabla v_i,$$

  i.e. Navier-Stokes is just Newton's second law in disguise.

Equations of motion

- Navier-Stokes equation:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} v = \rho g - \nabla p + \mu \Delta v$$

- SPH approximation:

$$\rho_i \frac{\mathrm{d}}{\mathrm{d}t} v_i = \rho_i g - (\nabla p)_i + \mu (\Delta v)_i$$

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- $\frac{\mu}{\rho_i} (\Delta v)_i = \frac{\mu}{\rho_i} \sum_{j=1}^{N} m_j \frac{v_j - v_i}{\rho_j} (\Delta W_h)(|r_i - r_j|)$

Pressure

Still need to compute pressure!

Pressure

- Ideal gas law:

$$p = k_B \frac{N}{V} T, \text{ where}$$

  - N number of molecules
  - V volume
  - T (absolute) temperature
  - k_B Boltzmann constant

Pressure

- Ideal gas law:

$$p = k_B \frac{N}{V} T$$

- Modelled as

$$p_i = k(\rho_i - \rho_{\text{eq}}), \text{ where}$$

  - k constant depending on temperature
  - ρ_{eq} equilibrium density (set to zero for ideal gas)

Boundary conditions

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  - Mixtures of Slip/Noslip
  - Pressure boundaries
  - Flux boundaries
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- In my implementation: Noslip solid boundaries using ghost particles.

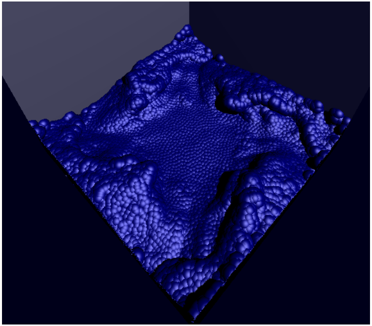
Advanced topics

Active areas of research in SPH include:

- Boundary modelling
- Adaptivity
- Surface tension
- Solid adhesion

Demo

- 64k particles, interactive frame-rates
- Graphics running against DirectX 11 (Windows only)
- Simulation running against OpenCL (Windows, Linux, Android, Supercomputers...)
- Surface tension and solid adhesion modelled according to Akinci, Akinci and Teschner (2013), Freiburg



Thanks for your attention!

Please do not hesitate to ask questions!