A Caccioppoli Inequality for Energy Minimizing Maps

Maximilian Wank

LMU München

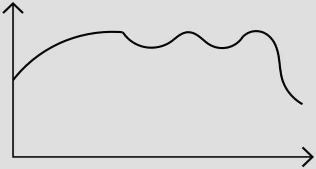
14.12.2013

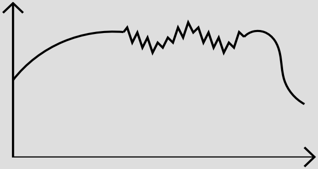
Bruck am Ziller

Poincaré vs. Caccioppoli – a comparison

Poincaré: $$\int_{B_\rho} \left| \frac{u - \langle u \rangle_{B_\rho}}{\rho} \right|^p \leq c \int_{B_\rho} |\nabla u|^p$$

Caccioppoli: $$\int_{B_\rho} |\nabla u|^p \leq c \int_{B_{2\rho}} \left| \frac{u - \langle u \rangle_{B_{2\rho}}}{2\rho} \right|^p$$





Simonenko Indices

Characterization of growth of N-functions:

$$p_0 := \inf_{t>0} \frac{\varphi'(t)t}{\varphi(t)} \leq \sup_{t>0} \frac{\varphi'(t)t}{\varphi(t)} =: p_1$$

- $\varphi(t) := t^p \implies p_0 = p = p_1$.
- Always assume restriction to $1 < p_0$ and $p_1 < \infty$.
- $\min\{s^{p_0}, s^{p_1}\}\varphi(t) \leq \varphi(st) \leq \max\{s^{p_0}, s^{p_1}\}\varphi(t)$

$$\implies \varphi(2t) \leq 2^{p_1}\varphi(t)$$

$$\implies \Delta_2 \text{ condition!}$$

Orlicz-Sobolev Spaces and Energy

Consider function spaces

$$W^{1,\varphi}(\Omega,\mathbb{R}^k) := \{u \in L^\varphi(\Omega,\mathbb{R}^k) \,:\, u \text{ weakly diff. and } |\nabla u| \in L^\varphi(\Omega)\},$$
$$W^{1,\varphi}(\Omega,M) := \{u \in W^{1,\varphi}(\Omega,\mathbb{R}^k) \,:\, u(x) \in M \text{ for a.e. } x \in \Omega\}.$$

For $B \Subset \Omega$ define local energy by

$$\mathcal{J}_B(u) := \int_B \varphi(|\nabla u|).$$

$u \in W^{1,\varphi}(\Omega, M)$ is energy minimizing iff $\forall B \Subset \Omega$ and $\forall v \in W^{1,\varphi}(\Omega, M)$ with $u \equiv v$ in a neighbourhood of ∂B

$$\mathcal{J}_B(u) \leq \mathcal{J}_B(v).$$

A Lemma of Luckhaus

M closed compact manifold, $\dim(M) = m$, N closed set in $\mathbb{R}^k$ and $p_1 - p_0 < 1$. Then, there are c and $S_0 \leq 1$ s.t. for all $u, v \in W^{1,\varphi}(M, N)$ and $0 < S < S_0$ there is a $w \in W^{1,\varphi}(M \times [0, S], \mathbb{R}^k)$ with

$$w(\,\cdot\,, 0) = u \text{ and } w(\,\cdot\,, S) = v.$$

Furthermore, for $\beta := \max\{n \in \mathbb{N} \,:\, n \leq p_0 - 1\}$ and $\delta > 0$ the estimates

$$\int\limits_{M\times(0,S)} \varphi(|\nabla w|) \leq cS^{1-(p_1-p_0)(m-\beta)} \underbrace{\int\limits_M \varphi(|\nabla u|) + \varphi(|\nabla v|) + \varphi\left(\frac{|u-v|}{S}\right)}_{:=A}$$

and

$$\varphi(\operatorname{dist}(w, N)) \leq cS^{\left(1-\frac{\beta}{p'}\right)p_0+\beta-m}\left(\delta^\alpha A + \delta^{\tilde{\alpha}} \int\limits_{M'} \varphi(|u-v|)\right).$$

A Lemma of Luckhaus – Sketch of Proof

- Slice M into submanifolds with lower dimension, i.e. M^β where $\beta := \max\{n \in \mathbb{N} : n \leq p_0 - 1\}$.
- Choose $w(x,s) := \frac{s}{S} v(x) + \frac{S-s}{S} u(x)$ for $x \in M^\beta$.
- Use $W^{1,\varphi}(M^\beta) \hookrightarrow W^{1,p_0}(M^\beta) \hookrightarrow C^{1-\frac{\beta}{p_0}}(\overline{M^\beta})$ since $\beta < p_0$ $\Longrightarrow L^\infty(M^\beta)$ estimate yields upper bound for $\varphi(\mathrm{dist}(w,N))$.
- Use radially homogeneous extension of degree zero from $\partial B_S^{\ell+1}$ to $B_S^{\ell+1}$ in parametrization of $M^{\ell+1}$.
- Upper bound for $\varphi(\mathrm{dist}(w,N))$ is preserved.

A Conclusion of Luckhaus' Lemma

Let N be a smooth compact manifold embedded in $\mathbb{R}^k$, $\Lambda > 0$ and $n \geq \frac{p_1}{p_0}$. There are $\delta_0 = \delta_0(n, N, \Lambda)$, $c = c(n, N, \Lambda)$ and $S_0 \leq 1$ such that for every $S \in (0, S_0)$ and $u \in W^{1,\varphi}(B_\rho, N)$ with

$$\rho^{-n} \int_{B_\rho} \varphi(\rho|\nabla u|) \leq \Lambda \text{ and } \rho^{-n} \int_{B_\rho} \varphi(|u - \langle u \rangle_{B_\rho}|) \leq \varphi(S^n \delta_0)$$

there is $\sigma \in (\frac{3\rho}{4}, \rho)$ such that there is a function $w = w_S \in W^{1,\varphi}(B_\rho, N)$ which agrees with u in a neighbourhood of $\partial B_\sigma(y)$ and which satisfies

$$\sigma^{-n} \int_{B_\sigma} \varphi(\sigma|\nabla w|) \leq c S^{1-(p_1-p_0)(m-\beta)} \left(\rho^{-n} \int_{B_\rho} \varphi(|\nabla u|) + \varphi\left(\frac{|u - \langle u \rangle_{B_\rho}|}{S} \right) \right).$$

Statement

Let u be energy minimizing and Λ be a constant. If

$$R^{p-n} \int_{B_R(x_0)} |\nabla u|^p \leq \Lambda$$

for some ball $B_R(x_0) \Subset \Omega$, then

$$\int_{B_\rho(y)} |\nabla u|^p \leq c \int_{B_{2\rho}(y)} \left| \frac{u - \langle u \rangle_{B_{2\rho(y)}}}{2\rho} \right|^p$$

for each $y \in B_{\frac{R}{2}}(x_0)$, $\rho < \frac{R}{8}$ and $c = c(n, \Lambda)$.